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Sylvester Matrix Equation for Matrix Pencils*

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ABSTRACT

We study a homogeneous linear matrix equation system related to the strict equivalence of matrix pencils. We obtain the dimension of the vector space of its solutions in terms of the invariants of the strict equivalence. We give a characterization of the strict equivalence of matrix pencils by rank tests, and we extend Roth's criterion for the corresponding nonhomogeneous system.

INTRODUCTION

In this paper we denote by $\mathbb{C}^{p\times q}$ the vector space over $\mathbb{C}$ of the $p \times q$ matrices with coefficients in $\mathbb{C}$, and by $\mathrm{Gl}_n(\mathbb{C})$ the linear group of degree n

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over $\mathbb{C}$. The ring of polynomials with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[\lambda]$, and the vector space over $\mathbb{C}$ of the $p \times q$ matrices with coefficients in $\mathbb{C}[\lambda]$ is denoted by $\mathbb{C}[\lambda]^{p\times q}$. Let $A, B \in \mathbb{C}^{n\times n}$. The Sylvester matrix equation $AX - XB = 0$ is closely related to the similarity relation between the matrices A and B: These matrices are similar if and only if this equation has an invertible solution. In the same way, two matrix pencils $\lambda B_1 - A_1$ and $\lambda B_2 - A_2$, where $B_1, A_1, B_2, A_2 \in \mathbb{C}^{p\times q}$, are strictly equivalent if and only if the generalized Sylvester matrix equation system

$$\begin{aligned} A_1 X - Y A_2 &= 0, \\ B_1 X - Y B_2 &= 0 \end{aligned} \tag{$*$}$$

has a solution (X, Y) with X and Y invertible matrices.

When $B_1, A_1 \in \mathbb{C}^{p_1\times q_1}$ and $B_2, A_2 \in \mathbb{C}^{p_2\times q_2}$, the dimension of the vector space of solutions of $(*)$ is obtained. The formula of this dimension is given in terms of the invariants for the strict equivalence of the pencils $\lambda B_i - A_i$, $i = 1, 2$. Instead of the Kronecker minimal indices we consider their conjugate partitions. The partitions corresponding to the common eigenvalues of these pencils (in $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$) in their Weyr characteristics appear too. As particular cases of this formula we find the formula for the classical Sylvester matrix equation $AX - XB = 0$ (Frobenius [5, 4]) and the formula for the Sylvester matrix equation associated with the block similarity (or feedback equivalence) relation between pairs of matrices [1, p. 97].

The obtained formula permits us to give a rational criterion of strict equivalence of singular matrix pencils by rank tests. This criterion extends a criterion for similarity of square matrices by rank tests given by S. Friedland [4] and a criterion for block similarity of pairs of matrices [1, p. 104].

We have worked with the matrix quadruple (A, B, C, D) associated with a linear time-invariant control system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned}$$

The transfer-function matrix $H(\lambda)$ of this system is given by $H(\lambda) = C(\lambda I - A)^{-1}B + D$. A singular pencil of matrices for this system can be constructed in the form

$$\begin{bmatrix} \lambda I - A & -B \\ -C & -D \end{bmatrix}.$$

An equivalence relation can be defined on the space of the quadruples (A, B, C, D), based on the basis changes in the state space, input space, and output space, and on the operations of state feedback and output injection. This equivalence relation is itself equivalent to the strict equivalence relation between the associated pencils to the quadruples. The minimal indices and elementary divisors are related to the system transfer-function matrix through a canonical form for the quadruples. There exists a minimal realization procedure based on the infinite elementary divisors. This theory was also applied to solve the problem of exact model matching (see [16] and [13]).

We have studied Equation $(*)$, reducing it to the analogous equation for a pair of matrix quadruples.

If $\lambda B_1 - A_1 \in \mathbb{C}[\lambda]^{q_1 \times q_1}$ and $\lambda B_2 - A_2 \in \mathbb{C}[\lambda]^{q_2 \times q_2}$ are regular matrix pencils without common eigenvalues (in $\overline{\mathbb{C}}$), then it has been proved that the linear operator

$$T : \mathbb{C}^{q_1 \times q_2} \times \mathbb{C}^{q_1 \times q_2} \to \mathbb{C}^{q_1 \times q_2} \times \mathbb{C}^{q_1 \times q_2},$$

$$(X, Y) \mapsto (A_1 X - YA_2, B_1 X - YB_2)$$

is invertible [15, p. 278, Theorem 1.11]. Our formula for the dimension of $\operatorname{Ker} T$ (Theorem 3.1) corroborates this result.

Besides the mathematical proofs of our results in Theorem 3.1, Lemmas 4.1, 4.2, 4.3, ad 4.4, and Theorem 4.5, we have checked these formulas by means of numerical examples using the MATLAB system in a 386 personal computer.

The paper ends with an extension of Roth's criterion to give a necessary and sufficient condition for the existence of a solution of a nonhomogeneous generalized Sylvester matrix equation system

$$A_1 X - YA_2 = A_3,$$

$$B_1 X - YB_2 = B_3$$

in terms of the strict equivalence of two adequate pencils.

The paper is structured as follows: Section 1 contains the classical definitions about pencils and matrix quadruples and their canonical forms. It also contains a simple formula for the dimension of the solution space of $AX - XB = 0$ by means of the scalar product of the partitions corresponding to the Weyr characteristics of the common eigenvalues of A and B. In Section 2 it is proved that our problem is stable with respect to the strict equivalence of pencils. Section 3 deals with the case of regular pencils.

Section 4 deals with the general singular pencils case through the matrix quadruples. Section 5 gives the extension of Roth's criterion.

1. PRELIMINARIES

1.1. *Partitions of Integers*

A *partition* is a finite or infinite sequence of nonincreasing nonnegative integers almost all zero,

$$m = (m_1, m_2, \dots).$$

The number of the nonzero terms m_i is the *Length* $\ell(m)$ of the partition m. The *conjugate partition* of m,

$$\overline{m} = (\overline{m}_1, \overline{m}_2, \dots)$$

is defined by

$$\overline{m}_k := \operatorname{Card}\{i : m_i \geq k\}.$$

If p is a positive integer, $\underline{p}$ denotes the set $\{1, 2, \dots, p\}$. The scalar product of $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ is defined by $x \cdot y = \sum_{k=1}^N x_k y_k$. Let $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ be partitions; the scalar product of a and b is defined by

$$a \cdot b = \sum_{i=1}^{N} a_i b_i,$$

where $N = \max\{\ell(a), \ell(b)\}$.

PROPOSITION 1.1. *Let* $m = (m_1, m_2, \dots, m_p, 0, \dots)$ *and* $n = (n_1, n_2, \dots, n_q, 0, \dots)$ *be partitions of lengths* p *and* q *respectively. Then*

$$\sum_{(i,j) \in \underline{p} \times \underline{q}} \min(m_i, n_j) = \overline{m} \cdot \overline{n}.$$

Proof. Let $\mu_{ij} := \min(m_i, n_j)$ for $(i, j) \in \underline{p} \times \underline{q}$. It is easy to prove by induction on t that

$$\sum_{(i,j) \in \underline{p} \times \underline{q}} \mu_{ij} = \sum_{k=1}^{t} \operatorname{Card}\{(i,j) \in \underline{p} \times \underline{q} \mid \mu_{ij} \geq k\} + \sum_{\{(i,j) : \mu_{ij} \geq t\}} (\mu_{ij} - t).$$

Suppose now that $m_1 \geq n_1$, without loss of generality; then putting $t = n_1$ in the preceding equality, it results that

$$\sum_{(i,j)\in \underline{p}\times\underline{q}} \mu_{ij} = \sum_{k=1}^{n_1} \operatorname{Card}\{(i,j) \in \underline{p} \times \underline{q} \mid \mu_{ij} \geq k\}$$

On the other hand,

$$\{(i,j) \in \underline{p} \times \underline{q} \mid \mu_{ij} \geq k\} = \{i \in \underline{p} \mid m_i \geq k\} \times \{j \in \underline{q} \mid n_j \geq k\}$$

So,

$$\sum_{(i,j)\in \underline{p}\times\underline{q}} \mu_{ij} = \sum_{k=1}^{n_1} \overline{m}_k \overline{n}_k. \qquad \blacksquare$$

Before seeing a consequence of this identity, we introduce some necessary notation. Let $M \in \mathbb{C}^{n\times n}$, and let the complex number λ_0 be an eigenvalue of M. We associate a partition to λ_0,

$$s(\lambda_0, M) = (m_1, m_2, \dots),$$

$m_1, m_2, \dots$ being the exponents of the elementary divisors of M associated to λ_0. As is known, the system of partitions $s(\lambda_0, M)$ corresponding to the different eigenvalues λ_0 of M is called the *Segrè characteristic* of M. If we define the conjugate partition

$$w(\lambda_0, M) = \overline{s(\lambda_0, M)},$$

the system of partitions $w(\lambda_0, M)$, when λ_0 ranges over the different eigenvalues of M, is called the *Weyr characteristic* of M. The spectrum of M is denoted by $\sigma(M)$.

Let now $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ such that $\sigma(A) \cap \sigma(B) = \{\lambda_1, \dots, \lambda_s\}$. Then the dimension $\nu(A, B)$ of the vector subspace $\{X \in \mathbb{C}^{m\times n} \mid AX - XB = 0\}$ of $\mathbb{C}^{m\times n}$ is equal to

$$\sum_{k=1}^{s} \sum_{(i,j)} \min(m_{ki}, n_{kj}),$$

where

$$(m_{k1}, m_{k2}, \dots) = s(\lambda_k, A), \quad (n_{k1}, n_{k2}, \dots) = s(\lambda_k, B)$$

$$(k = 1, \dots, s)$$

(see [4]). Then Proposition 1.1 implies that

$$\nu(A, B) = \sum_{k=1}^{s} w(\lambda_k, A) \cdot w(\lambda_k, B).$$

This formula corroborates the expression obtained in [2] for $\nu(A, A)$.

1.2. Strict Equivalence of Matrix Pencils

Let $H_1(\lambda) = \lambda B_1 - A_1$ and $H_2(\lambda) = \lambda B_2 - A_2$ be two matrix pencils where B_1, A_1, B_2, and A_2 are complex matrices of size $p \times q$. These pencils are said to be *strictly equivalent*, $H_1(\lambda) \overset{s}{\sim} H_2(\lambda)$, if there exist $P \in \mathrm{Gl}_p(\mathbb{C})$ and $Q \in \mathrm{Gl}_q(\mathbb{C})$ such that $PH_1(\lambda)Q = H_2(\lambda)$, that is to say,

$$PB_1Q = B_2 \quad \text{and} \quad PA_1Q = A_2. \tag{1.1}$$

A matrix pencil $H(\lambda) = \lambda B - A$ is said to be a *regular pencil* if A and B are square matrices and $\det(\lambda B - A)$ is not the zero polynomial. We define the *normal rank of the matrix pencil* $H(\lambda)$ as the order of its greatest minor different from the polynomial zero. We will denote it by $\operatorname{rkn} H$.

It is well known (see [7, Section (XII.5)] or [8, pp. 662–678]) that two pencils are strictly equivalent if and only if they have the same (column and row) minimal indices and the same (finite and infinite) elementary divisors. That is to say, a complete set of invariants for the relation of strict equivalence of pencils is formed by the following types of invariants:

(i) *column-minimal indices* denoted by $\varepsilon_1 \geq \cdots \geq \varepsilon_{r_1} > \varepsilon_{r_1+1} = \cdots = \varepsilon_{r_0} = 0$,

(ii) *row-minimal indices* denoted by $\eta_1 \geq \cdots \geq \eta_{s_1} > \eta_{s_1+1} = \cdots = \eta_{s_0} = 0$.

In (i) all the column-minimal indices may be equal to zero. In that case r_1 does not appear and the following developments will be simplified accordingly. The same is true for s_1 in (ii).

(iii) *infinite elementary divisors* of the form $\mu^{n_{\infty 1}}, \dots, \mu^{n_{\infty \nu_\infty}}$ with $n_{\infty 1} \geq \cdots \geq n_{\infty \nu_\infty} \geq 1$

(iv) *finite elementary divisors* of the form $(\lambda - \lambda_i)^{n_{i1}}, \dots, (\lambda - \lambda_i)^{n_{i\nu_i}}$ with $n_{i1} \geq \cdots \geq n_{i\nu_i} \geq 1$, $i = 1, \dots, u$.

We will also use the notation

$$s(\lambda_i, H(\lambda)) = (n_{i1}, \dots, n_{i\nu_i}, 0, \dots), \qquad i = 1, \dots, u,$$

$$s(\infty, H(\lambda)) = (n_{\infty 1}, \dots, n_{\infty \nu_\infty}, 0, \dots),$$

$[s(\lambda_1, H(\lambda)), \dots, s(\lambda_u, H(\lambda)), s(\infty, H(\lambda))]$ being the Segrè characteristic of the pencil $H(\lambda)$. The system of corresponding conjugate partitions

$$w(\lambda_i, H(\lambda)) = \overline{s(\lambda_i, H(\lambda))}, \qquad i = 1, \dots, u,$$

$$w(\infty, H(\lambda)) = \overline{s(\infty, H(\lambda))}$$

constitute the so-called Weyr characteristic of the pencil $H(\lambda)$. The spectrum of $H(\lambda)$ in $\overline{\mathbb{C}}$ is denoted by $\sigma(H(\lambda))$. It is known (see [9]) that $r_0 = q - \operatorname{rkn} H$ and $s_0 = p - \operatorname{rkn} H$.

Recall (see [7, 8]) that to each matrix pencil corresponds a canonical form called the *Kronecker canonical form*, which is determined, except for the order of the blocks, by the invariants described above.

For the particular case of regular pencils there are no minimal indices, and the corresponding canonical form, which is known as *Weierstrass canonical form*, is very simple. If $\sum_{i=1}^{u} \sum_{j=1}^{\nu_1} n_{ij} = n$ and $\sum_{j=1}^{\nu_\infty} n_{\infty j} = m$, the corresponding canonical form is

$$\lambda \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_m \end{bmatrix}, \tag{1.2}$$

where $N = \operatorname{diag}(N_1, \dots, N_{\nu_\infty})$, with

$$N_i = \begin{bmatrix} 0 & I_{n_{\infty i}-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{n_{\infty i} \times n_{\infty i}} \qquad \text{for} \quad i = 1, \dots, \nu_\infty,$$

and $J = \operatorname{diag}(J_{11}, \dots, J_{1\nu_1}, \dots, J_{u1}, \dots, J_{u\nu_u})$, with $J_{ij} = \lambda_i I_{n_{ij}} + N_{n_{ij}}$, with

$$N_{n_{ij}} = \begin{bmatrix} 0 & I_{n_{ij}-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{n_{ij} \times n_{ij}} \qquad \text{for} \quad j = 1, \dots, \nu_i, \quad i = 1, \dots, u.$$

1.3. *Equivalence of Matrix Quadruples*

Let $A_i \in \mathbb{C}^{n\times n}$, $B_i \in \mathbb{C}^{n\times m}$, $C_i \in \mathbb{C}^{p\times n}$, $D_i \in \mathbb{C}^{p\times m}$, $i = 1, 2$, and let us consider the quadruples (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2). These quadruples are said to be *equivalent* if there exist matrices $P \in \mathrm{Gl}_n(\mathbb{C})$, $Q \in \mathrm{Gl}_p(\mathbb{C})$, $T \in \mathrm{Gl}_m(\mathbb{C})$, $R \in \mathbb{C}^{n\times p}$, and $S \in \mathbb{C}^{m\times n}$ such that

$$\begin{bmatrix} P & R \\ 0 & Q \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ S & T \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}.$$

Matrix quadruples and matrix pencils are closely related in the sense that every quadruple (A, B, C, D) is associated with a pencil of the form

$$\lambda \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Conversely, for every pencil $H(\lambda) = \lambda H_1 - H_2 \in \mathbb{C}[\lambda]^{p\times q}$, if rank $H_1 = n$, there exist invertible matrices P and Q such that

$$P(\lambda H_1 - H_2)Q = \lambda \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

with $X_1 \in \mathbb{C}^{n\times n}$. We will say that the quadruple (X_1, X_2, X_3, X_4) is associated with the pencil $H(\lambda)$. A pencil may have different associated quadruples, but all of them will be equivalent, since it is known (see [9]) that the equivalence of matrix quadruples is a necessary and sufficient condition for the strict equivalence of the associated matrix pencils. As a consequence, we have that the invariants and the canonical form for the equivalence of quadruples will be obtained by means of the invariants and the Kronecker canonical form of the pencils that have them as associated quadruples. So, if the pencil

$$\lambda\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

has the invariants described in Section 1.2, the quadruple (A, B, C, D) is equivalent to the quadruple in canonical form (A_c, B_c, C_c, D_c) (see [9]). Here

$$A_c = \operatorname{diag}(A_\varepsilon, A_\eta, A_\infty, A_f),$$

with $A_\varepsilon = \operatorname{diag}(N_{\varepsilon_1}, \dots, N_{\varepsilon_{r_1}})$; $A_\eta = \operatorname{diag}(N_{\eta_1}^T, \dots, N_{\eta_{s_1}}^T)$; $A_\infty = \operatorname{diag}(N_{d_1}^T, \dots, N_{d_{t_1}}^T)$ with $d_i := n_{\infty i} - 1$, $d_1 \geq \cdots \geq d_{t_1} > d_{t_1+1} = \cdots = d_{t_0} = 0$, and $t_0 := \nu_\infty$; and $A_f = \operatorname{diag}(J_{11}, \dots, J_{i\nu_i}, \dots, J_{u1}, \dots, J_{u\nu_u})$, the N's and the J's being defined as in Section 1.2. Furthermore,

$$B_c = \begin{bmatrix} B_\varepsilon & 0 \\ 0 & 0 \\ 0 & B_\infty \\ 0 & 0 \end{bmatrix}$$

with

$$B_\varepsilon = \begin{bmatrix} E_1 \\ \vdots \\ E_{r_1} \end{bmatrix} \quad \text{and} \quad B_\infty = \begin{bmatrix} F_1 \\ \vdots \\ F_{t_1} \end{bmatrix},$$

where

$$E_{\varepsilon_j} = \begin{bmatrix} 0 \\ e_j \end{bmatrix} \in \mathbb{C}^{\varepsilon_j \times r_0}, \qquad F_j = \begin{bmatrix} e_j \\ 0 \end{bmatrix} \in \mathbb{C}^{d_j \times t_0},$$

and e_j is the row vector with 1 in the jth component and 0 in the rest. Next,

$$C_c = \begin{bmatrix} 0 & C_\eta & 0 & 0 \\ 0 & 0 & C_\infty & 0 \end{bmatrix},$$

where $C_\eta = [E_{\eta_1}^T, \dots, E_{\eta_{s_1}}^T]$, $E_{\eta_j}^T \in \mathbb{C}^{s_0 \times \eta_j}$; $C_\infty = [E_{d_1}^T, \dots, E_{d_{t_1}}^T]$ with $E_{d_j}^T \in$

$\mathbb{C}^{t_0 \times d_j}$; and $E_{\eta_j}^T$ and $E_{d_j}^T$ are the transposes of E_{η_j} and E_{d_j}, respectively, defined like E_{ε_j} above but with the sizes now required. Finally,

$$D_c = \begin{bmatrix} 0 & 0 \\ 0 & D_\infty \end{bmatrix} \in \mathbb{C}^{p \times m}, \qquad D_\infty = \begin{bmatrix} 0 & 0 \\ 0 & I_{t_0 - t_1} \end{bmatrix} \in \mathbb{C}^{t_0 \times t_0}.$$

1.4. *The* vec *Operator and the Kronecker Product*

We define the operator vec in the following way:

$$\text{vec:} \quad \mathbb{C}^{p \times q} \to \mathbb{C}^{pq \times 1}$$

$$X = (x_{ij}) \mapsto [x_{11}, \dots, x_{1q}, \dots, x_{p1}, \dots, x_{pq}]^T,$$

and we define the *Kronecker product* by

$$A \otimes B := (a_{ik} B) \in \mathbb{C}^{mp \times nq},$$

where $A = (a_{ik}) \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times q}$. It is known (see [12, p. 410]) that this operator and this product are related by the following property.

LEMMA 1.2. *If* $A \in \mathbb{C}^{m \times n}$, $X \in \mathbb{C}^{n \times p}$, *and* $B \in \mathbb{C}^{p \times q}$, *then*

$$\text{vec}(AXB) = (A \otimes B^T)\,\text{vec}(X).$$

DEFINITION. The *nullity* of a matrix $M \in \mathbb{C}^{m \times n}$, denoted by $\nu(M)$, is equal to n-rank M.

2. INVARIANCE OF THE DIMENSION BY STRICT EQUIVALENCE

Let us consider two matrix pencils $H_1(\lambda) = \lambda B_1 - A_1 \in \mathbb{C}[\lambda]^{p_1 \times q_1}$ and $H_2(\lambda) = \lambda B_2 - A_2 \in \mathbb{C}[\lambda]^{p_2 \times q_2}$ and the following matrix equation system:

$$\begin{aligned} A_1 X - Y A_2 &= 0, \\ B_1 X - Y B_2 &= 0. \end{aligned} \tag{2.1}$$

We will prove that the strict equivalence of pencils is a natural equivalence relation for the study of this system. That is to say, if $\lambda \tilde{B}_1 - \tilde{A}_1$ and $\lambda \tilde{B}_2 - \tilde{A}_2$ are two pencils strictly equivalent to the previous ones, respectively, then the vector space $\tilde{S}$ of solutions of the system (2.1) associated with these pencils is isomorphic to the analogous space S associated with $H_1(\lambda)$ and $H_2(\lambda)$.

PROPOSITION 2.1. *The vector spaces S and $\tilde{S}$ are isomorphic.*

Proof. By the strict equivalence of $\lambda B_i - A_i$ and $\lambda \tilde{B}_i - \tilde{A}_i$, there exist invertible matrices $P_i \in \mathrm{Gl}_{p_i}(\mathbb{C})$ and $Q_i \in \mathrm{Gl}_{q_i}(\mathbb{C})$ such that $P_i \tilde{B}_i Q_i = B_i$ and $P_i \tilde{A}_i Q_i = A_i$, $i = 1, 2$.

Let us consider the following map:

$$\varphi\colon\quad S \to \tilde{S}$$

$$(X, Y) \mapsto (\tilde{X}, \tilde{Y}) = \left(Q_1 X Q_2^{-1}, P_1^{-1} Y P_2\right).$$

It is easy to see that this linear map is an isomorphism. ∎

3. REGULAR PENCILS

3.1. *Dimension of the Solution Space*

We will study now the system (2.1) in the particular case when the pencils $H_1(\lambda)$ and $H_2(\lambda)$ are regular. By Proposition 2.1 we can suppose, without loss of generality, that $H_1(\lambda)$ and $H_2(\lambda)$ are in Weierstrass canonical form, so, if $q_1 \times q_1$ is the size of $H_i(\lambda)$, $i = 1, 2$, we can put

$$H_i(\lambda) = \lambda B_i - A_i = \lambda \begin{bmatrix} I_{n_i} & 0 \\ 0 & N_i \end{bmatrix} - \begin{bmatrix} J_i & 0 \\ 0 & I_{m_i} \end{bmatrix},$$

with n_i, m_i, J_i, N_i as the n, m, J, N in (1.2), corresponding to $H_i(\lambda)$, and $n_i + m_i = q_i$, $i = 1, 2$. Then the system (2.1) can be written

$$\begin{aligned} \begin{bmatrix} J_1 & 0 \\ 0 & I_{m_1} \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} - \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} J_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} &= 0, \\ \begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} - \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} I_{n_2} & 0 \\ 0 & N_2 \end{bmatrix} &= 0, \end{aligned} \tag{3.1}$$

where X_1, Y_1 have the size $n_1 \times n_2$, and X_4, Y_4 have the size $m_1 \times m_2$.

From (3.1) we can deduce the following matrix equations:

$$J_1X_1 - Y_1J_2 = 0, \tag{3.2}$$

$$J_1X_2 - Y_2 = 0, \tag{3.3}$$

$$X_3 - Y_3J_2 = 0, \tag{3.4}$$

$$X_4 - Y_4 = 0, \tag{3.5}$$

$$X_1 - Y_1 = 0, \tag{3.6}$$

$$X_2 - Y_2N_2 = 0, \tag{3.7}$$

$$N_1X_3 - Y_3 = 0, \tag{3.8}$$

$$N_1X_4 - Y_4N_2 = 0. \tag{3.9}$$

From (3.2) and (3.6) we obtain the equation

$$J_1X_1 - X_1J_2 = 0. \tag{3.10}$$

From (3.5) and (3.9) we obtain the equation

$$N_1X_4 - X_4N_2 = 0. \tag{3.11}$$

From (3.3) and (3.7) we obtain the equation

$$X_2 - J_1X_2N_2 = 0. \tag{3.12}$$

From (3.4) and (3.8) we obtain the equation

$$X_3 - N_1X_3J_2 = 0. \tag{3.13}$$

But the form of the solutions of (3.10) and (3.11) is well known (see [6, Section VIII.1]), and it is easy to see, too, that the equations (3.12) and (3.13) have only the zero solution. Moreover, if we denote by $\nu(J_1, J_2)$ and $\nu(N_1, N_2)$ the dimension of the solution vector spaces of (3.10) and (3.11) respectively, it is well known that these dimensions can be expressed in terms of the exponents of the (finite and infinite respectively) elementary divisors. So if $\mu^{n_{\infty 1}}, \dots, \mu^{n_{\infty \nu_\infty}}$ and $(\lambda - \lambda_1)^{n_{11}}, \dots, (\lambda - \lambda_1)^{n_{1\nu_1}}, \dots, (\lambda - \lambda_u)^{n_{u1}}, \dots, (\lambda - \lambda_u)^{n_{u\nu_u}}$ are the infinite and finite elementary divisors, respectively, of $H_1(\lambda)$, and if $\mu^{n'_{\infty 1}}, \dots, \mu^{n'_{\infty\rho_\infty}}$ and $(\lambda - \mu_1)^{n'_{11}}, \dots, (\lambda - \mu_1)^{n'_{1\rho_1}}, \dots, (\lambda - \mu_v)^{n'_{v1}}, \dots, (\lambda - \mu_v)^{n'_{v\rho_v}}$ are the infinite and finite elementary divisors, respectively, of $H_2(\lambda)$, then we have (see [6, 4])

$$\nu(J_1, J_2) = \sum_{i=1}^{u} \sum_{j=1}^{v} \sum_{k=1}^{\nu_1} \sum_{h=1}^{\rho_j} \delta_{ikjh},$$

where

$$\delta_{ikjh} := \deg\Big[\gcd\Big((\lambda - \lambda_i)^{n_{1k}}, (\lambda - \mu_j)^{n'_{jh}}\Big)\Big]$$

and

$$\nu(N_1, N_2) = \sum_{i=1}^{\nu_\infty} \sum_{j=1}^{\rho_\infty} \min(n_{\infty i}, n'_{\infty j}).$$

As we have said in Section 1.1, there are other possible expressions for $\nu(J_1, J_2)$ and $\nu(N_1, N_2)$.

So, if $H_i(\lambda)$, $i = 1, 2$, have the above invariants and S is the vector space of the solutions of the system (2.1) associated with $H_1(\lambda)$ and $H_2(\lambda)$, we can state:

THEOREM 3.1. *The dimension of the vector space S is*

$$\dim S = \sum_{k=1}^{s} w(\lambda_k, H_1(\lambda)) \cdot w(\lambda_k, H_2(\lambda)),$$

where $\sigma(H_1(\lambda)) \cap \sigma(H_2(\lambda)) = \{\lambda_1, \dots, \lambda_s\}$ *(in* $\overline{\mathbb{C}}$*).*

Proof. It is obvious, since $\dim S = \nu(J_1, J_2) + \nu(N_1, N_2)$ by the considerations before this theorem. ∎

3.2. *A Criterion for Strict Equivalence by Rank Tests*

Let $\lambda B_1 - A_1, \lambda B_2 - A_2 \in \mathbb{C}[\lambda]^{p\times q}$ be two pencils. As we have said in the Introduction, a possible way to determine whether these pencils are strictly equivalent is to consider the matrix equation system (2.1):

$$A_1 X - Y A_2 = 0,$$

$$B_1 X - Y B_2 = 0.$$

Then, $\lambda B_1 - A_1$ and $\lambda B_2 - A_2$ are strictly equivalent if and only if there exists a solution (X, Y) of (2.1) such that X and Y are invertible matrices.

Let now the pencils $H_i(\lambda) = \lambda B_i - A_i \in \mathbb{C}[\lambda]^{p_i \times q_i}$, $i = 1, 2$. By means of the vec operator, the Kronecker product, and Lemma 1.2, it is easy to see that the corresponding system (2.1) is equivalent to the linear system

$$\begin{bmatrix} A_1 \otimes I_{q_2} & -I_{p_1} \otimes A_2^T \\ B_1 \otimes I_{q_2} & -I_{p_1} \otimes B_2^T \end{bmatrix} \begin{bmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.14}$$

So the dimension of the vector space S of solutions of the system (2.1) coincides with the nullity of the coefficient matrix of (3.14), that is to say,

$$\dim S = \nu \begin{bmatrix} A_1 \otimes I_{q_2} & -I_{p_1} \otimes A_2^T \\ B_1 \otimes I_{q_2} & -I_{p_1} \otimes B_2^T \end{bmatrix}. \tag{3.15}$$

For the particular case of regular pencils, Theorem 3.1 gives an expression for the nullity of the corresponding coefficient matrix of (3.14). So we can establish now a criterion for strict equivalence of regular pencils which extends a criterion for similarity of square matrices due to Friedland (see [4]). In the proof of this criterion we will use the following result of Friedland:

LEMMA 3.2 [4]. *Let $A \in \mathbb{C}^{m \times m}$ and $B \in \mathbb{C}^{n \times n}$. Then*

$$\nu(A, B) \le \tfrac{1}{2}[\nu(A, A) + \nu(B, B)].$$

The equality holds if and only if $m = n$ and A and B are similar.

THEOREM 3.3. *The regular pencils $H_1(\lambda) = \lambda B_1 - A_1$ and $H_2(\lambda) = \lambda B_2 - A_2$ of size $q_i \times q_i$, $i = 1, 2$, are strictly equivalent if and only if the three matrices*

$$\begin{bmatrix} A_1 \otimes I_{q_1} & -I_{q_1} \otimes A_1^T \\ B_1 \otimes I_{q_1} & -I_{q_1} \otimes B_1^T \end{bmatrix}, \quad \begin{bmatrix} A_1 \otimes I_{q_2} & -I_{q_1} \otimes A_2^T \\ B_1 \otimes I_{q_2} & -I_{q_1} \otimes B_2^T \end{bmatrix},$$

$$\begin{bmatrix} A_2 \otimes I_{q_2} & -I_{q_2} \otimes A_2^T \\ B_2 \otimes I_{q_2} & -I_{q_2} \otimes B_2^T \end{bmatrix}$$

have the same rank.

Proof. By Proposition 2.1 and (3.15) the necessity of the condition is evident.

Conversely, if these three matrices have the same rank, then by (3.15) and Theorem 3.1 we have

$$\nu(J_1, J_1) + \nu(N_1, N_1) = \nu(J_1, J_2) + \nu(N_1, N_2)$$
$$= \nu(J_2, J_2) + \nu(N_2, N_2).$$

By Lemma 3.2 it holds that

$$\nu(J_1, J_2) \leq \tfrac{1}{2}[\nu(J_1, J_1) + \nu(J_2, J_2)] \tag{3.16}$$

and

$$\nu(N_1, N_2) \leq \tfrac{1}{2}[\nu(N_1, N_1) + \nu(N_2, N_2)]; \tag{3.17}$$

then

$$\nu(J_1, J_2) + \nu(N_1, N_2) \leq \tfrac{1}{2}[\nu(J_1, J_1) + \nu(N_1, N_1)$$
$$+ \nu(J_2, J_2) + \nu(N_2, N_2)],$$

and the equality holds if and only if we have equalities in (3.16) and (3.17). And then, Lemma 3.2 guarantees that J_1 and J_2 are similar and N_1 and N_2 are similar, and so $H_1(\lambda)$ and $H_2(\lambda)$ are strictly equivalent. ∎

4. SINGULAR PENCILS

4.1. A Matrix Equation for Matrix Quadruples

Let $H_i(\lambda) \in \mathbb{C}[\lambda]^{(n_i+p_i)\times(n_i+m_i)}$, $i = 1, 2$, be two matrix pencils. If n_i is the rank of the coefficient matrix of λ in $H_i(\lambda)$, $i = 1, 2$, then by transformations of strict equivalence in $H_i(\lambda)$, $i = 1, 2$, we can obtain the pencils

$$\lambda\begin{bmatrix} I_{n_i} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \qquad i = 1, 2.$$

Then the system (2.1) corresponding to these pencils is:

$$
\begin{aligned}
\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} - \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} &= 0, \\
\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} - \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}\begin{bmatrix} I_{n_2} & 0 \\ 0 & 0 \end{bmatrix} &= 0,
\end{aligned}
\tag{4.1}
$$

where $X_1, Y_1 \in \mathbb{C}^{n_1 \times n_2}$, $X_4 \in \mathbb{C}^{m_1 \times m_2}$, and $Y_4 \in \mathbb{C}^{p_1 \times p_2}$. From (4.1), it is easy to see that $X_1 = Y_1$, $X_2 = 0$, and $Y_3 = 0$. So we can consider the equivalent matrix equation related to the equivalence of matrix quadruples:

$$
\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix} - \begin{bmatrix} X_1 & Y_2 \\ 0 & Y_4 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = 0. \tag{4.2}
$$

Equations (4.1) and (4.2) have isomorphic spaces of solutions.

On the other hand, from (4.2) we can obtain the equations

$$
\begin{aligned}
A_1 X_1 + B_1 X_3 - X_1 A_2 - Y_2 C_2 &= 0, \\
B_1 X_4 - X_1 B_2 - Y_2 D_2 &= 0, \\
C_1 X_1 + D_1 X_3 - Y_4 C_2 &= 0, \\
D_1 X_4 - Y_4 D_2 &= 0.
\end{aligned}
\tag{4.3}
$$

By means of the vec operator, the Kronecker product, and Lemma 1.2, it is easy to see that (4.3) is equivalent to the linear system

$$
\begin{bmatrix}
A_1 \otimes I_{n_2} - I_{n_1} \otimes A_2^T & B_1 \otimes I_{n_2} & 0 & -I_{n_1} \otimes C_2^T & 0 \\
-I_{n_1} \otimes B_2^T & 0 & B_1 \otimes I_{m_2} & -I_{n_1} \otimes D_2^T & 0 \\
C_1 \otimes I_{n_2} & D_1 \otimes I_{n_2} & 0 & 0 & -I_{p_1} \otimes C_2^T \\
0 & 0 & D_1 \otimes I_{m_2} & 0 & -I_{p_1} \otimes D_2^T
\end{bmatrix}
$$

$$
\times \begin{bmatrix} \operatorname{vec}(X_1) \\ \operatorname{vec}(X_3) \\ \operatorname{vec}(X_4) \\ \operatorname{vec}(Y_2) \\ \operatorname{vec}(Y_4) \end{bmatrix} = 0. \tag{4.4}
$$

The dimension of the solution space of (4.1) or (4.2) coincides with the nullity of the coefficient matrix of (4.4).

4.2. *Dimension of the Solution Space*

From now on we will suppose the quadruples (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are in Kronecker canonical form, without loss of generality, so the form of these matrices is

$$A_i = \begin{bmatrix} A_\varepsilon^{(i)} & & & \\ & A_\eta^{(i)} & & \\ & & A_\infty^{(i)} & \\ & & & A_f^{(i)} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_\varepsilon^{(i)} & 0 \\ 0 & 0 \\ 0 & B_\infty^{(i)} \\ 0 & 0 \end{bmatrix},$$

$$C_i = \begin{bmatrix} 0 & C_\eta^{(i)} & 0 & 0 \\ 0 & 0 & C_\infty^{(i)} & 0 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0 & 0 \\ 0 & D_\infty^{(i)} \end{bmatrix}, \qquad i = 1, 2.$$

We want to obtain the rank of the coefficient matrix of (4.4). By permutations in the rows and the columns of the coefficient matrix of (4.4), we can obtain the equivalent matrix $\operatorname{diag}(M_1, M_2, M_3, M_4)$, where

$$M_1 = \begin{bmatrix} A_\varepsilon^{(1)} \otimes I_{n_2} - I_\alpha \otimes A_2^T & B_\varepsilon^{(1)} \otimes I_{n_2} & 0 & -I_\alpha \otimes C_2^T \\ -I_\alpha \otimes B_2^T & 0 & B_\varepsilon^{(1)} \otimes I_{m_2} & -I_\alpha \otimes D_2^T \end{bmatrix},$$

$$M_2 = \begin{bmatrix} A_\eta^{(1)} \otimes I_{n_2} - I_\beta \otimes A_2^T & 0 & -I_\beta \otimes C_2^T \\ -I_\beta \otimes B_2^T & 0 & -I_\beta \otimes D_2^T \\ C_\eta^{(1)} \otimes I_{n_2} & -I_{s_0} \otimes C_2^T & 0 \\ 0 & -I_{s_0} \otimes D_2^T & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} A_\infty^{(1)} \otimes I_{n_2} - I_\gamma \otimes A_2^T & B_\infty^{(1)} \otimes I_{n_2} & 0 & -I_\gamma \otimes C_2^T & 0 \\ -I_\gamma \otimes B_2^T & 0 & B_\infty^{(1)} \otimes I_{m_2} & -I_\gamma \otimes D_2^T & 0 \\ C_\infty^{(1)} \otimes I_{n_2} & D_\infty^{(1)} \otimes I_{n_2} & 0 & 0 & -I_{t_0} \otimes C_2^T \\ 0 & 0 & D_\infty^{(1)} \otimes I_{m_2} & 0 & -I_{t_0} \otimes D_2^T \end{bmatrix},$$

$$M_4 = \begin{bmatrix} A_f^{(1)} \otimes I_{n_2} - I_\delta \otimes A_2^T & -I_\delta \otimes C_2^T \\ -I_\delta \otimes B_2^T & -I_\delta \otimes D_2^T \end{bmatrix},$$

where $\alpha = \sum_{i=1}^{r_1} \varepsilon_i$, $\beta = \sum_{i=1}^{s_1} \eta_i$, $\gamma = \sum_{i=1}^{t_1} d_i$, $\delta = \sum_{i=1}^{u} \sum_{j=1}^{\nu_i} n_{ij}$, and where ε_i, $i = 1, \ldots, r_0$; η_i, $i = 1, \ldots, s_0$; d_i, $i = 1, \ldots, t_0$; and n_{ij}, $j = 1, \ldots, \nu_i$, $i = 1, \ldots, u$, are the invariants defined in Section 1.2 for the first quadruple (A_1, B_1, C_1, D_1). We will denote by ε'_i, $i = 1, \ldots, r'_0$; η'_i, $i = 1, \ldots, s'_0$; d'_i, $i = 1, \ldots, t'_0$; n'_{ij}, $j = 1, \ldots, \rho_i$, $i = 1, \ldots, v$, the corresponding invariants for the second quadrupole (A_2, B_2, C_2, D_2). Moreover, we will denote the conjugate partitions of $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{r_1})$, $\eta = (\eta_1, \ldots, \eta_{s_1})$, and $d = (d_1, \ldots, d_{t_1})$ by $r = (r_1, \ldots, r_{\varepsilon_1})$, $s = (s_1, \ldots, s_{\eta_1})$, and $t = (t_1, \ldots, t_{d_1})$, respectively (and similarly r', s', and t' for the conjugate partitions of ε', η', and d').

Now, we will obtain the nullity of these four matrices M_1, M_2, M_3, and M_4.

LEMMA 4.1. *The nullity of the matrix M_1 is*

$$\nu(M_1) = r_0(n_2 + m_2) + s'_0 \sum_{i=1}^{r_1} \varepsilon_i - \sum_{i=1}^{\varepsilon_1} r'_{i-1} r_i .$$

Proof. Taking into account the form of the matrices $A_\varepsilon^{(1)}$ and $B_\varepsilon^{(1)}$, by permutations of block rows and columns in M_1, we obtain the equivalent block-diagonal matrix:

$$\left[\begin{array}{ccc|c} M_1(\varepsilon_1) & & & \\ & \ddots & & 0 \\ & & M_1(\varepsilon_{r_1}) & \end{array}\right],$$

where

$$M_1(\varepsilon_i) = \left[\begin{array}{ccccc|c|c|ccccc} -A_2^T & I_{n_2} & \cdots & 0 & 0 & 0 & 0 & -C_2^T & & & & \\ 0 & -A_2^T & \cdots & 0 & 0 & 0 & 0 & & -C_2^T & & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & & \ddots & & \\ 0 & 0 & \cdots & -A_2^T & I_{n_2} & 0 & 0 & & & & -C_2^T & \\ 0 & 0 & \cdots & 0 & -A_2^T & I_{n_2} & 0 & & & & & -C_2^T \\ \hline -B_2^T & & & & & 0 & 0 & -D_2^T & & & & \\ & -B_2^T & & & & 0 & 0 & & -D_2^T & & & \\ & & \ddots & & & \vdots & \vdots & & & \ddots & & \\ & & & -B_2^T & & 0 & 0 & & & & -D_2^T & \\ & & & & -B_2^T & 0 & I_m & & & & & -D_2^T \end{array}\right]$$

and there are ε_i row blocks in each part of $M_1(\varepsilon_i)$, $i = 1, \ldots, r_1$.

Thus rank $M_1 = \sum_{i=1}^{r_1}$ rank $M_1(\varepsilon_i)$. So now we will obtain the rank of one block $M_1(\varepsilon_i)$.

By block column and block row transformations with the blocks I_{n_2} and I_{m_2} in $M_1(\varepsilon_i)$, we can obtain the equivalent matrix:

$$\left[\begin{array}{cccccc:c:c:ccccc}
0 & I_{n_2} & 0 & \cdots & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & I_{n_2} & \cdots & 0 & 0 & 0 & 0 & & & & & \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \multicolumn{5}{c}{\mathbf{0}} \\
0 & 0 & 0 & \cdots & 0 & I_{n_2} & 0 & 0 & & & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & I_{n_2} & 0 & & & & & \\
\hdashline
-B_2^T & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -D_2^T & 0 & \cdots & 0 & 0 \\
-B_2^T A_2^T & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -B_2^T C_2^T & -D_2^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\alpha_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \alpha_3 C_2^T & \alpha_4 C_2^T & \cdots & -D_2^T & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & I_{m_2} & 0 & 0 & \cdots & 0 & 0
\end{array}\right].$$

where $\alpha_n = -B_2^T(A_2^T)^{\varepsilon_i - n}$, $n = 2, 3, 4$.

Now, by permutations of block rows and columns we can obtain the equivalent matrix

$$\operatorname{diag}\left(I_{\varepsilon_i n_2 + m_2}, M_{1i}\right),$$

where

$$M_{1i} = \begin{bmatrix}
-B_2^T & -D_2^T & 0 & \cdots & 0 & 0 \\
-B_2^T A_2^T & -B_2^T C_2^T & -D_2^T & \cdots & 0 & 0 \\
-B_2^T\left(A_2^T\right)^2 & -B_2^T A_2^T C_2^T & -B_2^T C_2^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
-B_2^T\left(A_2^T\right)^{\varepsilon_i - 2} & -B_2^T\left(A_2^T\right)^{\varepsilon_i - 3} C_2^T & -B_2^T\left(A_2^T\right)^{\varepsilon_i - 4} C_2^T & \cdots & -D_2^T & 0
\end{bmatrix}.$$

So rank $M_1(\varepsilon_i) = \varepsilon_1 n_2 + m_2 +$ rank M_{1i}.

Now, taking into account the form of matrices A_2, B_2, C_2, and D_2, we can deduce that

$$B_2^T\left(A_2^T\right)^j = \begin{bmatrix}
B_\varepsilon^{(2)^T}\left(A_\varepsilon^{(2)^T}\right)^j & 0 & 0 & 0 \\
0 & 0 & B_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^j & 0
\end{bmatrix}, \qquad j = 0, \dots, \varepsilon_i - 2,$$

and

$$B_2^T\left(A_2^T\right)^j C_2^T = \begin{bmatrix} 0 & 0 \\ 0 & B_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^j C_\infty^{(2)^T} \end{bmatrix}, \qquad j = 0, \dots, \varepsilon_i - 3.$$

So, by block row and bock column permutations in M_{1i}, we can obtain the following matrix:

$$\begin{bmatrix} M_{1i\varepsilon} & 0 \\ 0 & M_{1i\infty} \end{bmatrix}, \tag{4.5}$$

where

$$M_{1i\varepsilon} := \begin{bmatrix} -B_\varepsilon^{(2)^T} \\ -B_\varepsilon^{(2)^T} A_\varepsilon^{(2)^T} \\ \vdots \\ -B_\varepsilon^{(2)^T}\left(A_\varepsilon^{(2)^T}\right)^{\varepsilon_i - 2} \end{bmatrix},$$

and

$$M_{1i\infty} := \begin{bmatrix} -B_\infty^{(2)^T} & -D_\infty^{(2)^T} & 0 & \cdots & 0 & 0 \\ -B_\infty^{(2)^T} A_\infty^{(2)^T} & -B_\infty^{(2)^T} C_\infty^{(2)^T} & -D_\infty^{(2)^T} & \cdots & 0 & 0 \\ -B_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^2 & -B_\infty^{(2)^T} A_\infty^{(2)^T} C_\infty^{(2)^T} & -B_\infty^{(2)^T} C_\infty^{(2)^T} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -B_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^{\varepsilon_i - 2} & -B_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^{\varepsilon_i - 3} C_\infty^{(2)^T} & -B_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^{\varepsilon_i - 4} C_\infty^{(2)^T} & \cdots & -D_\infty^{(2)^T} & 0 \end{bmatrix}$$

Therefore the rank of M_{1i} is equal to the sum of the ranks of the two blocks of the matrix (4.5). But it is known that

$$\operatorname{rank}\left[B_\varepsilon^{(2)}, A_\varepsilon^{(2)} B_\varepsilon^{(2)}, \dots, \left(A_\varepsilon^{(2)}\right)^{\varepsilon_i - 2} B_\varepsilon^{(2)}\right] = r_1' + \cdots + r_{\varepsilon_i - 1}',$$

because the pair $(A_{\varepsilon}^{(2)}, B_{\varepsilon}^{(2)})$ is in Brunovsky canonical form, its controllability indices are $\varepsilon'_1 \geq \cdots \geq \varepsilon'_{r'_1} > 0$, and its Brunovsky numbers are $r'_1 \geq \cdots \geq r'_{\varepsilon'_1} > 0$ (see [1; 8, p. 196]). So we only need to obtain the rank of the lower block of matrix (4.5).

Now taking into account the form of matrices $A_{\infty}^{(2)}$, $B_{\infty}^{(2)}$, $C_{\infty}^{(2)}$, and $D_{\infty}^{(2)}$, if we put

$$B_{\infty}^{(2)^T} = \left[F_1^{(2)^T}, \ldots, F_{t'_1}^{(2)^T}\right] = \begin{bmatrix} F_{\infty}^{(2)^T} \\ 0 \end{bmatrix} \qquad \text{with} \quad F_{\infty}^{(2)^T} \in \mathbb{C}^{t'_1 \times \sum_{j=1}^{t'_1} d'_j},$$

$$C_{\infty}^{(2)^T} = \begin{bmatrix} E_{d'_1}^{(2)} \\ \vdots \\ E_{d'_{t'_1}}^{(2)} \end{bmatrix} = \left[E_{\infty}^{(2)}, 0\right] \qquad \text{with} \quad E_{\infty}^{(2)} \in \mathbb{C}^{\sum_{j=1}^{t'_1} d'_j \times t'_1},$$

we can decompose the blocks of the lower block of (4.5) in the following way:

$$\begin{bmatrix}
-F_{\infty}^{(2)^T} & 0 & 0 & 0 & 0 & \cdots 0 & 0 & 0 \\
0 & 0 & I_{t'_0-t'_1} & 0 & 0 & \cdots 0 & 0 & 0 \\
-F_{\infty}^{(2)^T} A_{\infty}^{(2)^T} & -F_{\infty}^{(2)^T} E_{\infty}^{(2)} & 0 & 0 & 0 & \cdots 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{t'_0-t'_1} & \cdots 0 & 0 & 0 \\
-F_{\infty}^{(2)^T}\left(A_{\infty}^{(2)^T}\right)^2 & -F_{\infty}^{(2)^T} A_{\infty}^{(2)^T} E_{\infty}^{(2)} & 0 & -F_{\infty}^{(2)^T} E_{\infty}^{(2)} & 0 & \cdots 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-F_{\infty}^{(2)^T}\left(A_{\infty}^{(2)^T}\right)^{\varepsilon_i-2} & -F_{\infty}^{(2)^T}\left(A_{\infty}^{(2)^T}\right)^{\varepsilon_i-3} E_{\infty}^{(2)} & 0 & -F_{\infty}^{(2)^T}\left(A_{\infty}^{(2)^T}\right)^{\varepsilon_i-4} E_{\infty}^{(2)} & 0 & \cdots 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots 0 & I_{t'_0-t'_1} & 0
\end{bmatrix}. \tag{4.6}$$

By permutations in the rows and the columns of (4.6) we obtain the equivalent matrix

$$\left[\begin{array}{c|ccccc} I_{(\varepsilon_i-1)(t'_0-t'_1)} & & & & & \\ \hline & -F_\infty^{(2)^T} & 0 & \cdots & 0 & 0 \\ & -F_\infty^{(2)^T}A_\infty^{(2)^T} & -F_\infty^{(2)^T}E_\infty^{(2)} & \cdots & 0 & 0 \\ & -F_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^2 & -F_\infty^{(2)^T}A_\infty^{(2)^T}E_\infty^{(2)} & \cdots & 0 & 0 \\ & \vdots & \vdots & & \vdots & \vdots \\ & -F_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^{\varepsilon_i-2} & -F_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^{\varepsilon_i-3}E_\infty^{(2)} & \cdots & -F_\infty^{(2)^T}E_\infty^{(2)} & 0 \end{array}\right]. \tag{4.7}$$

But, by the form of matrices $F_\infty^{(2)}$, $A_\infty^{(2)}$, and $E_\infty^{(2)}$, we can deduce that the rank of the lower block of (4.7) is

$$\begin{aligned} &\operatorname{rank} F_\infty^{(2)^T} + \operatorname{rank}\left[F_\infty^{(2)^T}A_\infty^{(2)^T}, F_\infty^{(2)^T}E_\infty^{(2)}\right] + \cdots \\ &\quad + \operatorname{rank}\left[F_\infty^{(2)^T}\left(A_\infty^{(2)^T}\right)^{\varepsilon_i-2}, \dots, F_\infty^{(2)^T}E_\infty^{(2)}\right] \\ &\quad = t'_1 + \left(\operatorname{Card}\{i : d'_i \geqslant 2\} + \operatorname{Card}\{i : d'_i = 1\}\right) + \cdots \\ &\qquad + \left(\operatorname{Card}\{i : d'_i \geqslant \varepsilon_i - 1\} + \operatorname{Card}\{i : d'_i = \varepsilon_i - 2\} + \cdots \right. \\ &\qquad\qquad \left. + \operatorname{Card}\{i : d'_i = 1\}\right) \\ &\quad = t'_1 + t'_1 + \cdots + t'_1 = (\varepsilon_i - 1)t'_1. \end{aligned}$$

So

$$\operatorname{rank} M_{1i} = \sum_{j=1}^{\varepsilon_i-1} r'_j + (\varepsilon_i - 1)(t'_0 - t'_1) + (\varepsilon_i - 1)t'_1 = \sum_{j=1}^{\varepsilon_i-1} r'_j + (\varepsilon_i - 1)t'_0$$

and

$$\operatorname{rank} M_1(\varepsilon_i) = \varepsilon_i n_2 + m_2 + \sum_{j=1}^{\varepsilon_i-1} r'_j + (\varepsilon_i - 1)t'_0.$$

Therefore,

$$\operatorname{rank} M_1 = \sum_{i=1}^{r_1} \operatorname{rank} M_1(\varepsilon_i) = r_1 m_2 + n_2 \sum_{i=1}^{r_1} \varepsilon_i + \sum_{i=1}^{r_1} \sum_{j=1}^{\varepsilon_i - 1} r'_j + t'_0 \sum_{i=1}^{r_1} (\varepsilon_i - 1).$$

Since the number of columns of M_1 is

$$n_2 \sum_{i=1}^{r_1} \varepsilon_i + r_0 n_2 + r_0 m_2 + (s'_0 + t'_0) \sum_{i=1}^{r_1} \varepsilon_i,$$

we have

$$\nu(M_1) = r_0(n_2 + m_2) + s'_0 \sum_{i=1}^{r_1} \varepsilon_i - r_1 m_2 - \sum_{i=1}^{r_1} \sum_{j=1}^{\varepsilon_i - 1} r'_j + t'_0 r_1.$$

Since $m_2 = r'_0 + t'_0$, we can put

$$\nu(M_1) = r_0(n_2 + m_2) + s'_0 \sum_{i=1}^{r_1} \varepsilon_i - r_1 r'_0 - \sum_{i=1}^{r_1} \sum_{j=1}^{\varepsilon_i - 1} r'_j.$$

But it is known (see [1, p. 102]) that

$$\sum_{i=1}^{r_1} \sum_{j=1}^{\varepsilon_i - 1} r'_j = \sum_{i=2}^{\varepsilon_1} r'_{i-1} r_i.$$

Therefore,

$$\nu(M_1) = r_0(n_2 + m_2) + s'_0 \sum_{i=1}^{r_1} \varepsilon_i - \sum_{i=1}^{\varepsilon_1} r'_{i-1} r_i. \qquad \blacksquare$$

LEMMA 4.2. *The nullity of the matrix* M_2 *is*

$$\nu(M_2) = s_0 s'_0 + s'_0 \sum_{i=1}^{s_1} \eta_i - \sum_{i=0}^{\eta_1} s'_{i+1} s_i.$$

Proof. As in Lemma 4.1, taking into account the form of the matrices $A_\eta^{(1)}$ and $C_\eta^{(1)}$, by permutations of block rows and columns in M_2, we obtain the equivalent block-diagonal matrix:

$$\begin{bmatrix} M_2(\eta_1) & & & \\ & \ddots & & \\ & & M_2(\eta_{s_1}) & \\ & & & M_2(s_0 - s_1) \end{bmatrix},$$

where

$$M_2(\eta_i) = \left[\begin{array}{ccccc|c|ccccc}
-A_2^T & 0 & \cdots & 0 & 0 & & -C_2^T & & & & \\
I_{n_2} & -A_2^T & \cdots & 0 & 0 & & & -C_2^T & & & \\
\vdots & \vdots & & \vdots & \vdots & 0 & & & \ddots & & \\
0 & 0 & \cdots & -A_2^T & 0 & & & & & -C_2^T & \\
0 & 0 & \cdots & I_{n_2} & -A_2^T & & & & & & -C_2^T \\
\hline
-B_2^T & & & & & & -D_2^T & & & & \\
 & -B_2^T & & & & & & -D_2^T & & & \\
 & & \ddots & & & 0 & & & \ddots & & \\
 & & & -B_2^T & & & & & & -D_2^T & \\
 & & & & -B_2^T & & & & & & -D_2^T \\
\hline
0 & 0 & \cdots & 0 & I_{n_2} & -C_2^T & & & 0 & & \\
\hline
 & & 0 & & & -D_2^T & & & 0 & &
\end{array}\right],$$

in which the blocks A_2^T and B_2^T appear η_i times, $i = 1, \ldots, s_1$; and where

$$M_2(s_0 - s_1) = \left[\begin{array}{ccc}
-C_2^T & & \\
 & \ddots & \\
 & & -C_2^T \\
\hline
-D_2^T & & \\
 & \ddots & \\
 & & -D_2^T
\end{array}\right],$$

in which the blocks C_2^T and D_2^T appear $s_0 - s_1$ times. Thus,

$$\operatorname{rank} M_2 = \sum_{i=1}^{s_1} \operatorname{rank} M_2(\eta_i) + \operatorname{rank} M_2(s_0 - s_1).$$

But rank $M_2(s_0 - s_1) = (s_0 - s_1)(s_1' + t_0')$. So now we obtain the rank of a block $M_2(\eta_i)$.

By permutations in the rows and the columns of $M_2(\eta_i)$ we can obtain the equivalent matrix

$$\left[\begin{array}{c:c:c:c:c:c}
\begin{matrix} -A_2^T & -C_2^T \\ -B_2^T & -D_2^T \end{matrix} & 0 & 0 & \cdots & 0 & 0 \\ \hdashline
\begin{matrix} I_{n_2} & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -A_2^T & -C_2^T \\ -B_2^T & -D_2^T \end{matrix} & 0 & \cdots & 0 & 0 \\ \hdashline
0 & \begin{matrix} I_{n_2} & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -A_2^T & -C_2^T \\ -B_2^T & -D_2^T \end{matrix} & \cdots & 0 & 0 \\ \hdashline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hdashline
0 & 0 & 0 & \cdots & \begin{matrix} -A_2^T & -C_2^T \\ -B_2^T & -D_2^T \end{matrix} & 0 \\ \hdashline
0 & 0 & 0 & \cdots & \begin{matrix} I_{n_2} & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -C_2^T \\ -D_2^T \end{matrix}
\end{array}\right]. \tag{4.8}$$

Now, taking into account the form of the matrices A_2, B_2, C_2, and D_2, we can put the submatrix

$$\left[\begin{array}{cc}
-A_2^T & -C_2^T \\
-B_2^T & -D_2^T \\ \hdashline
I_{n_2} & 0
\end{array}\right]$$

in the form

$$
\left[\begin{array}{c|c|c|c|cc}
-A_{\varepsilon}^{(2)^T} & & & & \multicolumn{2}{c}{\mathbf{0}} \\
\hline
 & -N_{\eta'_1} & & & -E^{(2)}_{\eta'_1} & 0 \\
 & \ddots & & & \vdots & \vdots \\
 & -N_{\eta'_{s'_1}} & & & -E^{(2)}_{\eta'_{s'_1}} & 0 \\
\hline
 & & -N_{d'_1} & & 0 & -E^{(2)}_{d'_1} \\
 & & \ddots & & \vdots & \vdots \\
 & & -N_{d'_{t'_1}} & & 0 & -E^{(2)}_{d'_{t'_1}} \\
\hline
 & & & -A_f^{(2)^T} & \multicolumn{2}{c}{\mathbf{0}} \\
\hline
-B_{\varepsilon}^{(2)^T} & & 0 \quad \cdots \quad 0 & & 0 & 0 \\
 & \mathbf{0} & & \mathbf{0} & & \\
0 & & -F_1^{(2)^T} \quad \cdots \quad -F_{t'_1}^{(2)^T} & & 0 & -D_{\infty}^{(2)} \\
\hline
I_{\alpha'} & & & & & \\
 & I_{\eta'_1} & & & & \\
 & \ddots & & & & \\
 & I_{\eta'_{s'_1}} & & & \multicolumn{2}{c}{\mathbf{0}} \\
 & & I_{d'_1} & & & \\
 & & \ddots & & & \\
 & & I_{d'_{t'_1}} & & & \\
 & & & I_{\delta'} & &
\end{array}\right]
\tag{4.9}
$$

where $\alpha' = \sum_{i=1}^{r'_1} \varepsilon'_i$ and $\delta' = \sum_{i=1}^{v} \sum_{j=1}^{\rho_i} n'_{ij}$.

Now, we will perform column transformations on the matrix (4.8) from the right to the left. If we denote

$$
I_{\beta'} = \operatorname{diag}\left(I_{\eta'_1}, \dots, I_{\eta'_{s'_1}} \right) \quad \text{and} \quad I_{\gamma'} = \operatorname{diag}\left(I_{d'_1}, \dots, I_{d'_{t'_1}} \right),
$$

with the first $s'_1 + t'_1$ nonzero columns of C_2^T at the right of (4.8), we can put zeros in rows $\eta'_1, \eta'_1 + \eta'_2, \dots, \eta'_1 + \cdots + \eta'_{s'_1}$ of $I_{\beta'}$ and in rows $d'_1, d'_1 + d'_2, \dots, d'_1 + \cdots + d'_{t'_1}$ of $I_{\gamma'}$. So in columns $\alpha' + \eta'_1, \alpha' + \eta'_1 + \eta'_2, \dots, \alpha' + \eta'_1 + \cdots + \eta'_{s'_1}$ and $\alpha' + \beta' + d'_1, \alpha' + \beta' + d'_1 + d'_2, \dots, \alpha' + \beta' + d'_1 + \cdots + d'_{t'_1}$ of the block (4.9) there will remain only a 1 in the position $(\eta'_i - 1, \eta'_i)$ of the corresponding $N_{\eta'_i}$ if $\eta'_i > 1$, and in the position $(d'_j -$

$1, d'_j)$ of the corresponding $N_{d'_j}$ if $d'_j > 1$. Moreover, for the $N_{\eta'_i}$ such that $\eta'_i = 1$ the corresponding columns in (4.8) are zero. So the rank of the last but one block of (4.8) after these transformations is

$$s'_1 + t'_0 + n_2 - \operatorname{Card}\{j : \eta'_j = 1\} = t'_0 + n_2 + s'_2$$

because of

$$\operatorname{Card}\{j : \eta'_j = 1\} = \operatorname{Card}\{j : \eta'_j \geqslant 1\} - \operatorname{Card}\{j : \eta'_j \geq 2\} = s'_1 - s'_2.$$

Now, we will perform the same kind of transformations on the previous block of type (4.9) with the columns of A_2^T mentioned above that are not zero columns and the first $s'_1 + t'_1$ nonzero columns of C_2^T in order to put some zeros in the I_{n_2}. These zeros will be in rows

$$\eta'_1 - 1, \eta'_1, \eta'_1 + \eta'_2 - 1, \eta'_1 + \eta'_2, \dots, \eta'_1 + \cdots + \eta'_{s'_2} - 1, \eta'_1 + \cdots + \eta'_{s'_2},$$
$$\eta'_1 + \cdots + \eta'_{s'_2+1}, \dots, \eta'_1 + \cdots + \eta'_{s'_1}$$

of $I_{\beta'}$, and in rows

$$d'_1 - 1, d'_1, d'_1 + d'_2 - 1, d'_1 + d'_2, \dots, d'_1 + \cdots + d'_{t'_2} - 1, d'_1 + \cdots + d'_{t'_2},$$
$$d'_1 + \cdots + d'_{t'_2+1}, \dots, d'_1 + \cdots + d'_{t'_1}$$

of $I_{\gamma'}$. So, in columns

$$\alpha' + \eta'_1, \alpha' + \eta'_1 + \eta'_2, \dots, \alpha' + \eta'_1 + \cdots + \eta'_{s'_2}$$

and

$$\alpha' + \beta' + d'_1, \alpha' + \beta' + d'_1 + d'_2, \dots, \alpha' + \beta' + d'_1 + \cdots + d'_{t'_2}$$

of the block (4.9) there will remain only a 1 in the position $(\eta'_i - 1, \eta'_i)$ of the corresponding $N_{\eta'_i}$ if $\eta'_i > 1$. Moreover, for the $N_{\eta'_i}$ such that $\eta'_i = 1$ the corresponding column in (4.8) is zero. There also remains a 1 in the position $(d'_j - 1, d'_j)$ Of the corresponding $N_{d'_j}$ if $d'_j > 1$. In columns $\alpha' + \eta'_1 - 1, \alpha' + \eta'_1 + \eta'_2 - 1, \dots, \alpha' + \eta'_1 + \cdots + \eta'_{s'_3} - 1$ and $\alpha' + \beta' + d'_1 - 1, \alpha' + \beta' + d'_1 + d'_2 - 1, \dots, \alpha' + \beta' + d'_1 + \cdots + d'_{t'_3} - 1$ there remains only a 1 in the position $(\eta'_i - 2, \eta'_i - 1)$ of the corresponding $N_{\eta'_i}$ if $\eta'_i > 2$. Column

$\alpha' + \eta_1' + \cdots + \eta_i' - 1$ in (4.9) is zero if $\eta_i' = 2$. And there remains only a 1 in the column of (4.8) in the position $(d_j' - 2, d_j' - 1)$ of the corresponding $N_{d_j'}$ if $d_j' > 2$.

Thus, the rank of this block after these transformation is

$$s_1' + t_0' + n_2 - \operatorname{Card}\{j : \eta_j' = 1\} - \operatorname{Card}\{j : \eta_j' = 2\} = t_0' + n_2 + s_3'.$$

We can proceed in the same way until the first block of (4.8). The rank of this first block before performing the same kind of transformations will be

$$t_0' + n_2 + s_{\eta_i+1}'.$$

Therefore, the rank of the matrix $M_2(\eta_i)$ will be

$$\operatorname{rank} M_2(\eta_i) = s_1' + t_0' + \sum_{j=1}^{\eta_i} \left(t_0' + n_2 + s_{j+1}'\right),$$

and we have

$$\operatorname{rank} M_2 = s_1(s_1' + t_0') + \sum_{i=1}^{s_1} \sum_{j=1}^{\eta_i} \left(t_0' + n_2 + s_{j+1}'\right) + (s_0 - s_1)(s_1' + t_0')$$

$$= s_0(s_1' + t_0') + (t_0' + n_2) \sum_{i=1}^{s_1} \eta_i + \sum_{i=1}^{s_1} \sum_{j=1}^{\eta_i} s_{j+1}'.$$

Since the number of columns of M_2 is

$$n_2 \sum_{i=1}^{s_1} \eta_i + s_0 p_2 + p_2 \sum_{i=1}^{s_1} \eta_i, \qquad \text{where} \quad p_2 = s_0' + t_0',$$

we have

$$\nu(M_2) = s_0(s_0' - s_1') + s_0' \sum_{i=1}^{s_1} \eta_i - \sum_{i=1}^{s_1} \sum_{j=1}^{\eta_i} s_{j+1}'$$

$$= s_0 s_0' + s_0' \sum_{i=1}^{s_1} \eta_i - \sum_{i=0}^{\eta_1} s_{i+1}' s_i. \qquad \blacksquare$$

REMARK. Observe that $\sum_{i=0}^{\eta_1} s'_{i+1} s_i = \sum_{i=1}^{\eta'_1} s_{i-1} s'_i$.

LEMMA 4.3. *The nullity of the matrix* M_3 *is*

$$\nu(M_3) = t_0 s'_0 + s'_0 \sum_{i=1}^{t_1} d_i + \sum_{i=0}^{d_1} t_i t'_i .$$

Proof. Taking into account the form of the matrices $A_\infty^{(1)}$, $B_\infty^{(1)}$, $C_\infty^{(1)}$, and $D_\infty^{(1)}$, by permutations of block rows and columns in M_3 we obtain the equivalent block-diagonal matrix:

$$\begin{bmatrix} M_3(d_1) & & & \\ & \ddots & & \\ & & M_3(d_{t_1}) & \\ & & & M_3(t_0 - t_1) \end{bmatrix},$$

where

$$M_3(d_1) = \left[\begin{array}{ccccc|c|c|cccc|c}
-A_2^T & 0 & \cdots & 0 & 0 & I_{n_2} & 0 & -C_2^T & & & & \\
I_{n_2} & -A_2^T & \cdots & 0 & 0 & 0 & 0 & & -C_2^T & & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & & \ddots & & 0 \\
0 & 0 & \cdots & -A_2^T & 0 & 0 & 0 & & & -C_2^T & & \\
0 & 0 & \cdots & I_{n_2} & -A_2^T & 0 & 0 & & & & -C_2^T & \\
\hline
-B_2^T & & & & & 0 & I_{m_2} & -D_2^T & & & & \\
& -B_2^T & & & & 0 & 0 & & -D_2^T & & & \\
& & \ddots & & & \vdots & \vdots & & & \ddots & & 0 \\
& & & -B_2^T & & 0 & 0 & & & -D_2^T & & \\
& & & & -B_2^T & 0 & 0 & & & & -D_2^T & \\
\hline
0 & 0 & \cdots & 0 & I_{n_2} & 0 & 0 & & & 0 & & -C_2^T \\
\hline
& & 0 & & & 0 & 0 & & & 0 & & -D_2^T
\end{array}\right]$$

in which the blocks A_2^T and B_2^T appear d_i times, $i = 1, \dots, t_1$; and where

$$M_3(t_0 - t_1) = \left[\begin{array}{ccc|ccc|ccc}
I_{n_2} & & & & & & -C_2^T & & \\
& \ddots & & & & & & \ddots & \\
& & I_{n_2} & & & & & & -C_2^T \\ \hline
& & & I_{m_2} & & & -D_2^T & & \\
& & & & \ddots & & & \ddots & \\
& & & & & I_{m_2} & & & -D_2^T
\end{array}\right],$$

in which the blocks C_2^T and D_2^T appear $t_0 - t_1$ times. Thus,

$$\operatorname{rank} M_3 = \sum_{i=1}^{t_1} \operatorname{rank} M_3(d_i) + \operatorname{rank} M_3(t_0 - t_1).$$

But $\operatorname{rank} M_3(t_0 - t_1) = (t_0 - t_1)(n_2 + m_2)$. So now we obtain the rank of a block $M_3(d_i)$.

With the blocks I_{n_2} and I_{m_2} of $M_3(d_i)$, which are alone in their block column, we can cancel the corresponding row block. So, after permutations in block rows and columns, we can obtain the equivalent matrix

$$\left[\begin{array}{c|c|cc|cc|c|cc|c}
I_{n_2+m_2} & 0 & 0 & & 0 & & \cdots & 0 & & 0 \\ \hline
0 & I_{n_2} & -A_2^T & -C_2^T & 0 & & \cdots & 0 & & 0 \\
& 0 & -B_2^T & -D_2^T & & & & & & \\ \hline
0 & 0 & I_{n_2} & 0 & -A_2^T & -C_2^T & \cdots & 0 & & 0 \\
& & 0 & 0 & -B_2^T & -D_2^T & & & & \\ \hline
\vdots & \vdots & \vdots & & \vdots & & \ddots & \vdots & & \vdots \\ \hline
0 & 0 & 0 & & 0 & & \cdots & -A_2^T & -C_2^T & 0 \\
& & & & & & & -B_2^T & -D_2^T & \\ \hline
0 & 0 & 0 & & 0 & & \cdots & -I_{n_2} & 0 & -C_2^T \\
& & & & & & & 0 & 0 & -D_2^T
\end{array}\right], \tag{4.10}$$

where the block

$$\begin{bmatrix} -A_2^T & -C_2^T \\ -B_2^T & -D_2^T \end{bmatrix}$$

appears $d_i - 1$ times.

Now, we will perform column and row transformations on the matrix (4.10) from the left to the right. So, with the first block I_{n_2} on the left, we will cancel the matrices A_2^T and C_2^T in their block row. Then, with the 1's of the matrix B_2^T we cancel some 1's of the second block I_{n_2} on the left, by row transformations. More precisely, we can put zeros in the following rows of I_{n_2}:

$$\varepsilon'_1, \varepsilon'_1 + \varepsilon'_2, \dots, \varepsilon'_1 + \cdots + \varepsilon'_{r'_1},$$

$$\alpha' + \beta' + 1, \alpha' + \beta' + d'_1 + 1, \dots, \alpha' + \beta' + d'_1 + \cdots + d'_{t'_1 - 1} + 1.$$

If we denote by $I_{n_2}^{(1)}$ this matrix with zeros in those positions, then by column transformations with $I_{n_2}^{(1)}$ on the following block column, we can put zeros in all the positions of the matrices A_2^T and C_2^T except for the positions $(\varepsilon'_i, \varepsilon'_i - 1)$ of the blocks $N_{\varepsilon'_i}^T$, $i = 1, \dots, r'_1$, if $\varepsilon'_i \geq 2$; the positions $(1, 2)$ of the blocks $N_{d'_i}$, $i = 1, \dots, t'_1$, if $d'_i \geq 2$; and the positions $(\alpha' + \beta' + \sum_{i=1}^{j} d'_i, s'_0 + j)$, $j = 1, \dots, t'_1$ of C_2^T, if $d'_j = 1$. Now, with the 1's of the matrix B_2^T in the fourth block column, and the 1's which remain in the A_2^T of the same block column, by row transformations we can put zeros in the following rows of the submatrix $I_{\alpha'} = \operatorname{diag}(I_{\varepsilon'_1}, \dots, I_{\varepsilon'_{r'_1}})$ of I_{n_2}:

$$\varepsilon'_1 - 1, \varepsilon'_1, \varepsilon'_1 + \varepsilon'_2 - 1, \varepsilon'_1 + \varepsilon'_2, \dots, \varepsilon'_1 + \cdots + \varepsilon'_{r'_2} - 1,$$

$$\varepsilon'_1 + \cdots + \varepsilon'_{r'_2}, \varepsilon'_1 + \cdots + \varepsilon'_{r'_2 + 1}, \dots, \varepsilon'_1 + \cdots + \varepsilon'_{r'_1};$$

and we can put zeros in the following positions of the submatrix $I_{\gamma'} = \operatorname{diag}(I_{d'_1}, \dots, I_{d'_{t'_1}})$ of I_{n_2}: $(1, 1)$ of $I_{d'_i}$ for every $i = 1, \dots, t'_1$, and $(2, 2)$ of $I_{d'_i}$ if $d'_i \geq 2$.

If we denote now by $I_{n_2}^{(2)}$ this matrix with zeros in those positions, then by column transformations with $I_{n_2}^{(2)}$ on the following block column, we can put zeros in all the positions of the matrices A_2^T and C_2^T except for the positions $(\varepsilon'_i - 1, \varepsilon'_i - 2)$, of the blocks $N_{\varepsilon'_i}^T$, $i = 1, \dots, r'_1$, if $\varepsilon'_i \geq 3$; the positions $(\varepsilon'_i, \varepsilon'_i - 1)$ of the same blocks if $\varepsilon'_i \geq 2$; the positions $(1, 2)$ of the blocks $N_{d'_i}$, $i = 1, \dots, t'_1$, if $d'_i \geq 2$; the positions $(2, 3)$ of the same blocks if $d'_i \geq 3$; and the positions $(\alpha' + \beta' + \sum_{i=1}^{j} d'_i, s'_0 + j)$, $j = 1, \dots, t'_1$, of C_2^T if $d'_j \leq 2$.

We continue in the same way until the last block column. Now we will compute the ranks of the block columns which remain after performing these transformations on the matrix (4.10). The sum of these ranks is

$$\begin{aligned}
&(n_2 + m_2) + n_2 + (n_2 + t_0' - t_1') + \left(n_2 + \operatorname{Card}\{j : d_j' = 1\} + t_0' - t_1'\right)\\
&\qquad + \left(n_2 + \operatorname{Card}\{j : d_j' \le 2\} + t_0' - t_1'\right) + \cdots\\
&\qquad + \left(n_2 + \operatorname{Card}\{j : d_j' \le d_i - 2\} + t_0' - t_1'\right)\\
&\qquad + \left(\operatorname{Card}\{j : d_j' \le d_i - 1\} + t_0' - t_1'\right)\\
&\quad = n_2 + m_2 + n_2 d_1 + (t_0' - t_1')d_i + \operatorname{Card}\{j : d_j' = 1\}\\
&\qquad + \operatorname{Card}\{j : d_j' \le 2\} + \cdots + \operatorname{Card}\{j : d_j' \le d_i - 1\}.
\end{aligned}$$

But, by the definition of conjugate partitions, we can see that $\operatorname{Card}\{j : d_j' \le i\} = t_1' - t_{i+1}'$. So

$$\begin{aligned}
\operatorname{rank} M_3(d_i) &= n_2 + m_2 + n_2 d_i + (t_0' - t_1')d_i + \sum_{j=1}^{d_i} (t_1' - t_j')\\
&= n_2 + m_2 + (n_2 + t_0')d_i - \sum_{j=1}^{d_i} t_j'.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{rank} M_3 &= t_1(n_2 + m_2) + (n_2 + t_0') \sum_{i=1}^{t_1} d_i - \sum_{i=1}^{t_1} \sum_{j=1}^{d_i} t_j' + (t_0 - t_1)(n_1 + m_2)\\
&= t_0(n_2 + m_2) + (n_2 + t_0') \sum_{i=1}^{t_1} d_i - \sum_{i=1}^{t_1} \sum_{j=1}^{d_i} t_j'.
\end{aligned}$$

Since the number of columns of M_3 is

$$n_2 \sum_{i=1}^{t_1} d_i + t_0(n_2 + m_2) + (s_0' + t_0') \sum_{i=1}^{t_1} d_i + t_0(s_0' + t_0'),$$

we have

$$\nu(M_3) = s'_0 \sum_{i=1}^{t_1} d_i + t_0(s'_0 + t'_0) + \sum_{i=1}^{t_1} \sum_{j=1}^{d_i} t'_j$$

$$= t_0 s'_0 + s'_0 \sum_{i=1}^{t_1} d_i + \sum_{i=0}^{d_1} t_i t'_i. \qquad \blacksquare$$

LEMMA 4.4. *The nullity of the matrix* M_4 *is*

$$\nu(M_4) = s'_0 \sum_{i=1}^{u} \sum_{k=1}^{\nu_i} n_{ik} + \sum_{i=1}^{u} \sum_{j=1}^{v} \sum_{k=1}^{\nu_i} \sum_{h=1}^{\rho_j} \delta_{ikjh},$$

where $\delta_{ikjh} = \deg[\gcd((\lambda - \lambda_i)^{n_{ik}}, (\lambda - \mu_j)^{n'_{jh}})]$.

Proof. Taking into account the form of the matrix $A_f^{(1)}$, by permutations of block rows and columns in M_4, we obtain the equivalent block-diagonal matrix:

$$\begin{bmatrix} M_4(n_{11}) & & & & & & \\ & \ddots & & & & & \\ & & M_4(n_{1\nu_1}) & & & & \\ & & & \ddots & & & \\ & & & & M_4(n_{u1}) & & \\ & & & & & \ddots & \\ & & & & & & M_4(n_{u\nu_u}) \end{bmatrix},$$

where

$$M_4(n_{ik}) = \left[\begin{array}{ccccc:ccccc}
\lambda_i I_{n_2} - A_2^T & I_{n_2} & \cdots & 0 & 0 & -C_2^T & & & & \\
0 & \lambda_i I_{n_2} - A_2^T & \cdots & 0 & 0 & & -C_2^T & & & \\
\vdots & \vdots & & \vdots & \vdots & & & \ddots & & \\
0 & 0 & \cdots & \lambda_i I_{n_2} - A_2^T & I_{n_2} & & & & -C_2^T & \\
0 & 0 & \cdots & 0 & \lambda_i I_{n_2} - A_2^T & & & & & -C_2^T \\
\hdashline
-B_2^T & & & & & -D_2^T & & & & \\
 & -B_2^T & & & & & -D_2^T & & & \\
 & & \ddots & & & & & \ddots & & \\
 & & & -B_2^T & & & & & -D_2^T & \\
 & & & & -B_2^T & & & & & -D_2^T
\end{array}\right].$$

and the blocks A_2^T, B_2^T, C_2^T, and D_2^T appear n_{ik} times, $k = 1, \ldots, \nu_i$, $i = 1, \ldots, u$.

Now, taking into account the form of the matrices A_2^T, B_2^T, C_2^T, and D_2^T, by permutations of block columns and rows in $M_4(n_{ik})$, we can obtain the equivalent block diagonal matrix

$$\operatorname{diag}(M_{41}(n_{ik}), M_{42}(n_{ik}), M_{43}(n_{ik}), M_{44}(n_{ik})),$$

where

$$M_{41}(n_{ik}) = \left[\begin{array}{ccccc}
\lambda_i I_{\alpha'} - A_\varepsilon^{(2)^T} & I_{\alpha'} & \cdots & 0 & 0 \\
0 & \lambda_i I_{\alpha'} - A_\varepsilon^{(2)^T} & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & \lambda_i I_{\alpha'} - A_\varepsilon^{(2)^T} & I_{\alpha'} \\
0 & 0 & \cdots & 0 & \lambda_i I_{\alpha'} - A_\varepsilon^{(2)^T} \\
\hdashline
-B_\varepsilon^{(2)^T} & & & & \\
 & -B_\varepsilon^{(2)^T} & & & \\
 & & \ddots & & \\
 & & & -B_\varepsilon^{(2)^T} & \\
 & & & & -B_\varepsilon^{(2)^T}
\end{array}\right],$$

$$M_{42}(n_{ik}) = \left[\begin{array}{cccc:cccc}
\lambda_i I_{\beta'} - A_\eta^{(2)^T} & I_{\beta'} & \cdots & 0 & -C_\eta^{(2)^T} & & & \\
0 & \lambda_i I_{\beta'} - A_\eta^{(2)^T} & \cdots & 0 & & -C_\eta^{(2)^T} & & \\
\vdots & \vdots & & \vdots & & & \ddots & \\
0 & 0 & \cdots & I_{\beta'} & & & -C_\eta^{(2)^T} & \\
0 & 0 & \cdots & \lambda_i I_{\beta'} - A_\eta^{(2)^T} & & & & -C_\eta^{(2)^T}
\end{array}\right]$$

$$M_{43}(n_{ik}) = \left[\begin{array}{cccc:cccc} \lambda_i I_{\gamma'} - A_\infty^{(2)^T} & I_{\gamma'} & \cdots & 0 & -C_\infty^{(2)^T} & & & \\ 0 & \lambda_i I_{\gamma'} - A_\infty^{(2)^T} & \cdots & 0 & & -C_\infty^{(2)^T} & & \\ \vdots & \vdots & & \vdots & & & \ddots & \\ 0 & 0 & \cdots & \lambda_i I_{\gamma'} - A_\infty^{(2)^T} & & & & -C_\infty^{(2)^T} \\ \hdashline -B_\infty^{(2)^T} & & & & -D_\infty^{(2)^T} & & & \\ & -B_\infty^{(2)^T} & & & & -D_\infty^{(2)^T} & & \\ & & \ddots & & & & \ddots & \\ & & & -B_\infty^{(2)^T} & & & & -D_\infty^{(2)^T} \end{array}\right],$$

$$M_{44} = (n_{ik}) = \begin{bmatrix} \lambda_i I_{\delta'} - A_f^{(2)^T} & I_{\delta'} & \cdots & 0 & 0 \\ 0 & \lambda_i I_{\delta'} - A_f^{(2)^T} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_i I_{\delta'} - A_f^{(2)^T} & I_{\delta'} \\ 0 & 0 & \cdots & 0 & \lambda_i I_{\delta'} - A_f^{(2)^T} \end{bmatrix},$$

and the matrices $A_\varepsilon^{(2)^T}$, $A_\eta^{(2)^T}$, $A_\infty^{(2)^T}$, and $A_f^{(2)^T}$ appear n_{ik} times in their blocks $M_{41}(n_{ik})$, $M_{42}(n_{ik})$, $M_{43}(n_{ik})$, and $M_{44}(n_{ik})$, $k = 1, \dots, \nu_i$, $i = 1, \dots, u$.

Now we obtain the nullities of these blocks. The columns of $M_{41}(n_{ik})$ are linearly independent, so

$$\nu\big(M_{41}(n_{ik})\big) = 0.$$

Analogously, the rows of $M_{42}(n_{ik})$ are linearly independent, so

$$\nu\big(M_{42}(n_{ik})\big) = n_{ik} s_0'.$$

By the form of matrices $A_\infty^{(2)}$, $B_\infty^{(2)}$, $C_\infty^{(2)}$, and $D_\infty^{(2)}$ it is not difficult to see that the matrix $M_{43}(n_{ik})$ is regular. So

$$\nu\big(M_{43}(n_{ik})\big) = 0.$$

Finally, it is known (see [4]) that

$$\nu\big(M_{44}(n_{ik})\big) = \sum_{j=1}^{v} \sum_{h=1}^{\rho_j} \delta_{ikjh}.$$

Then

$$\nu\big(M_4(n_{ik})\big) = s_0' n_{ik} + \sum_{j=1}^{v} \sum_{h=1}^{\rho_j} \delta_{ikjh}$$

and

$$\nu(M_4) = s'_0 \sum_{i=1}^{u} \sum_{k=1}^{\nu_i} n_{ik} + \sum_{i=1}^{u} \sum_{j=1}^{v} \sum_{k=1}^{\nu_i} \sum_{h=1}^{\rho_j} \delta_{ikjh}. \qquad \blacksquare$$

Now, we can establish the dimension of the solution vector space of the system (2.1) associated with two singular pencils $H_1(\lambda)$ and $H_2(\lambda)$. If $H_i(\lambda) \in \mathbb{C}[\lambda]^{(n_i+p_i)\times(n_i+m_i)}$, $i = 1, 2$, are two pencils with invariants as above, and S is the solution vector space of the system (2.1) associated with them, we can state:

THEOREM 4.5. *The dimension of the vector space S is*

$$\dim S = r_0(n_2 + m_2) + s'_0(n_1 + p_1) - \sum_{i=1}^{\varepsilon_1} r'_{i-1} r_i - \sum_{i=1}^{\eta'_1} s_{i-1} s'_i$$

$$+ \sum_{k=1}^{s} w(\lambda_k, H_1(\lambda)) \cdot w(\lambda_k, H_2(\lambda)),$$

where $\{\lambda_1, \dots, \lambda_s\} = \sigma(H_1(\lambda)) \cap \sigma(H_2(\lambda))$ *(in* $\overline{\mathbb{C}}$*).*

Proof. By Section 4.1 and Lemmas 4.1, 4.2, 4.3, and 4.4, we have that

$$\dim S = r_0(n_2 + m_2) + s'_0(n_1 + p_1) - \sum_{i=1}^{\varepsilon_1} r'_{i-1} r_i - \sum_{i=1}^{\eta'_1} s_{i-1} s'_i$$

$$+ \sum_{i=0}^{d_1} t'_i t_i + \sum_{i=1}^{u} \sum_{j=1}^{v} \sum_{k=1}^{\nu_i} \sum_{h=1}^{\rho_j} \delta_{ikjh}.$$

By Proposition 1.1, we can observe that

$$\sum_{i=0}^{d_1} t_i t'_i = w(\infty, H_1(\lambda)) \cdot w(\infty, H_2(\lambda)) = \sum_{i=1}^{t_0} \sum_{j=1}^{t'_0} \min(n_{\infty i}, n'_{\infty j}).$$

So the two last terms of the above expression can be written

$$\sum_{k=1}^{s} w(\lambda_k, H_1(\lambda)) \cdot w(\lambda_k, H_2(\lambda)). \qquad \blacksquare$$

4.3. A Criterion for Strict Equivalence by Rank Tests

As in Section 3.2, we are going to give a criterion for strict equivalence of two singular pencils by rank tests. In this case, as in [1] for a criterion for block similarity of matrix pairs, we must add to the equality of the ranks of three matrices as in Theorem 3.3 the conditions that the minimal indices are to be the same in the two pencils. We are going to use a known result (see [11, Corollary 3.2; 14, Corollary 1; 10, Theorems 1.7, 1.8, pp. 99–100]) which characterizes the conjugate partitions of the minimal indices in terms of the nullities of some matrices. If $H(\lambda) = \lambda B - A \in \mathbb{C}[\lambda]^{p\times q}$, we define for $k = 1, 2, \dots$

$$T_k(H) = T_k(B, A) = \begin{bmatrix} B & 0 & \cdots & 0 \\ -A & B & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B \\ 0 & 0 & \cdots & -A \end{bmatrix} \in \mathbb{C}^{(k+1)p\times kq}.$$

With the notation for the minimal indices as above, we can state the following result:

LEMMA 4.6. *Let $H(\lambda) = \lambda B - A \in \mathbb{C}[\lambda]^{p\times q}$. Then for $k = 1, 2, \dots$ we have*:

(i) $\nu(T_k(B, A)) = kr_0 - \sum_{i=1}^{k} r_i$,
(ii) $\nu(T_k(B^T, A^T)) = ks_0 - \sum_{i=1}^{k} s_i$,

where $r_0 = q - \operatorname{rkn} H$, $s_0 = p - \operatorname{rkn} H$, $r_i = \operatorname{Card}\{j : \varepsilon_j \ge i\}$, *and* $s_i = \operatorname{Card}\{j : \eta_j \ge i\}$, $i = 1, 2, \dots, k$.

Now, we can give the following criterion for strict equivalence of matrix pencils:

THEOREM 4.7. *The pencils $H_1(\lambda) = \lambda B_1 - A_1$ and $H_2(\lambda) = \lambda B_2 - A_2$ of size $p \times q$ are strictly equivalent if and only if*:

$$\text{(i) }\operatorname{rank}\begin{bmatrix} A_1 \otimes I_q & -I_p \otimes A_1^T \\ B_1 \otimes I_q & -I_p \otimes B_1^T \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1 \otimes I_q & -I_p \otimes A_2^T \\ B_1 \otimes I_q & -I_p \otimes B_2^T \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_2 \otimes I_q & -I_p \otimes A_2^T \\ B_2 \otimes I_q & -I_p \otimes B_2^T \end{bmatrix},$$

(ii) $\operatorname{rkn} H_1 = \operatorname{rkn} H_2$,

(iii) $\operatorname{rank} T_k(H_1) = \operatorname{rank} T_k(H_2)$, $k = 1, 2, \ldots, p$,
(iv) $\operatorname{rank} T_k(H_1^T) = \operatorname{rank} T_k(H_2^T)$, $k = 1, 2, \ldots, q$.

Proof. By Proposition 2.1 and (3.15) the necessity of condition (i) is evident. By Lemma 4.6 the necessity of conditions (ii), (iii), and (iv) is evident too.

Conversely, suppose that (i), (ii), (iii), and (iv) hold. From (ii), (iii), and (iv) we have

$$r_i = r_i', \quad i = 0, 1, \ldots, p, \quad \text{and} \quad s_i = s_i', \quad i = 0, 1, \ldots, q;$$

that is to say, $H_1(\lambda)$ and $H_2(\lambda)$ have the same column and row minimal indices.

Now, from (i)–(iv), Theorem 4.5, and Theorem 3.3, we can deduce that the pencils $H_1(\lambda)$ and $H_2(\lambda)$ have the same finite and infinite elementary divisors. Therefore, $H_1(\lambda)$ and $H_2(\lambda)$ are strictly equivalent.

Analogously, if we denote by $\mathbb{M}_{ij}$ the coefficient matrix of (4.4) associated with the quadruples (A_i, B_i, C_i, D_i) and (A_j, B_j, C_j, D_j), $i, j = 1, 2$, we can state a criterion for equivalence of matrix quadruples as follows:

COROLLARY 4.8. *Let* $(A_i, B_i, C_i, D_i) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{p\times n} \times \mathbb{C}^{p\times m}$, $i = 1, 2$. *Then these quadruples are equivalent if and only if*:

(i) $\operatorname{rank} \mathbb{M}_{11} = \operatorname{rank} \mathbb{M}_{12} = \operatorname{rank} \mathbb{M}_{22}$
(ii) $\operatorname{rkn} H_1 = \operatorname{rkn} H_2$.
(iii) $\operatorname{rank} T_k(H_1) = \operatorname{rank} T_k(H_2)$, $k = 1, \ldots, m$,
(iv) $\operatorname{rank} T_k(H_1^T) = \operatorname{rank} T_k(H_2^T)$, $k = 1, \ldots, p$,

where

$$H_i(\lambda) = \lambda \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \qquad i = 1, 2.$$

5. EXTENSION OF ROTH'S CRITERION

We will consider now the nonhomogeneous system associated with the system (2.1):

$$\begin{aligned} A_1 X - YA_2 &= A_3, \\ B_1 X - YB_2 &= B_3, \end{aligned} \tag{5.1}$$

where $A_1, B_1 \in \mathbb{C}^{p_1 \times q_1}$, $A_2, B_2 \in \mathbb{C}^{p_2 \times q_2}$, $A_3, B_3 \in \mathbb{C}^{p_1 \times q_2}$, $X \in \mathbb{C}^{q_1 \times q_2}$, and $Y \in \mathbb{C}^{p_1 \times p_2}$. In this section we will give a criterion for the solvability of (5.1) in terms of the strict equivalence of the pencils

$$H_1(\lambda) = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \lambda - \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and}$$

$$H_2(\lambda) = \begin{bmatrix} B_1 & B_3 \\ 0 & B_2 \end{bmatrix} \lambda - \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}.$$

THEOREM 5.1. *The system* (5.1) *has a solution if and only if the pencils* $H_1(\lambda)$ *and* $H_2(\lambda)$ *are strictly equivalent.*

Proof. The proof of this theorem is inspired by the proof of Roth's criterion made by H. Flanders and H. K. Wimmer [3].

If (5.1) has a solution (X, Y), then if we define

$$P = \begin{bmatrix} I_{p_1} & -Y \\ 0 & I_{p_2} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} I_{q_1} & X \\ 0 & I_{q_2} \end{bmatrix},$$

we have that P and Q are invertible matrices and $PH_1(\lambda)Q = H_2(\lambda)$. So $H_1(\lambda)$ and $H_2(\lambda)$ are strictly equivalent.

Conversely, if $H_1(\lambda)$ and $H_2(\lambda)$ are strictly equivalent pencils, there exist $P \in \mathrm{Gl}_{p_1+p_2}(\mathbb{C})$ and $Q \in \mathrm{Gl}_{q_1 \times q_2}(\mathbb{C})$ such that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = P \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} Q \quad \text{and} \quad \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} = P \begin{bmatrix} B_1 & B_3 \\ 0 & B_2 \end{bmatrix} Q.$$

Now, if we put $m = p_1 + p_2$ and $n = q_1 + q_2$, we define the linear transformations T_1, T_2 in the following way:

$$T_1: \quad \mathbb{C}^{n \times n} \times \mathbb{C}^{m \times m} \to \mathbb{C}^{m \times n} \times \mathbb{C}^{m \times n},$$

$$(X, Y) \mapsto \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} X - Y \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} X - Y \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \right);$$

$$T_2:\quad \mathbb{C}^{n\times n}\times\mathbb{C}^{m\times m}\to\mathbb{C}^{m\times n}\times\mathbb{C}^{m\times n},$$

$$(X,Y)\mapsto\left(\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}X-Y\begin{bmatrix}A_1 & A_3\\ 0 & A_2\end{bmatrix},\right.$$

$$\left.\begin{bmatrix}B_1 & 0\\ 0 & B_2\end{bmatrix}X-Y\begin{bmatrix}B_1 & B_3\\ 0 & B_2\end{bmatrix}\right).$$

Let $R_i=\operatorname{Ker}T_i$, $i=1,2$. Since the pencils $H_1(\lambda)$ and $H_2(\lambda)$ are strictly equivalent, by Theorem 4.7 we have that

$$\dim R_1=\dim R_2. \tag{5.2}$$

Now, we are going to characterize the matrices in R_1 and R_2. If we decompose the matrices $X\in\mathbb{C}^{n\times n}$ and $Y\in\mathbb{C}^{m\times m}$ in blocks

$$X=\begin{bmatrix}X_1 & X_2\\ X_3 & X_4\end{bmatrix}\quad\text{and}\quad Y=\begin{bmatrix}Y_1 & Y_2\\ Y_3 & Y_4\end{bmatrix},$$

where $X_1\in\mathbb{C}^{q_1\times q_1}$, $X_4\in\mathbb{C}^{q_2\times q_2}$, $Y_1\in\mathbb{C}^{p_1\times p_1}$, and $Y_4\in\mathbb{C}^{p_2\times p_2}$, we have that R_1 is the set of pairs

$$\left(\begin{bmatrix}X_1 & X_2\\ X_3 & X_4\end{bmatrix},\begin{bmatrix}Y_1 & Y_2\\ Y_3 & Y_4\end{bmatrix}\right)$$

which satisfy the conditions

(1a) $A_1X_1-Y_1A_1=0$,
(2a) $A_1X_2-Y_2A_2=0$,
(3a) $A_2X_3-Y_3A_1=0$,
(4a) $A_2X_4-Y_4A_2=0$,
(5a) $B_1X_1-Y_1B_1=0$,
(6a) $B_1X_2-Y_2B_2=0$,
(7a) $B_2X_3-Y_3B_1=0$,
(8a) $B_2X_4-Y_4B_2=0$.

Analogously, R_2 is the set of pairs (X,Y) that satisfy conditions (1a), (3a), (5a), (7a), and

(2b) $A_1X_2-Y_2A_2=Y_1A_3$,
(4b) $A_2X_4-Y_4A_2=Y_3A_3$,
(6b) $B_1X_2-Y_2B_2=Y_1B_3$,
(8b) $B_2X_4-Y_4B_2=Y_3B_3$.

Taking into account conditions (2b) and (6b), we observe that it is sufficient to find a pair in R_2 such that $Y_1 = I_{p_1}$ in order to guarantee the existence of a solution of (5.1).

Let us consider now the following subspace of $\mathbb{C}^{n\times q_1} \times \mathbb{C}^{m\times p_1}$:

$$\mathscr{S} := \left\{ \left(\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix} \right) \text{ satisfying (1a), (3a), (5a), and (7a)} \right\}.$$

We define the transformations $\varphi_i : R_i \to \mathscr{S}$, $i = 1, 2$, in the following way:

$$\varphi_i \left(\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \right) = \left(\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix} \right).$$

It is easy to see that

$$\begin{aligned} \operatorname{Ker} \varphi_1 &= \operatorname{Ker} \varphi_2 \\ &= \left\{ \left(\begin{bmatrix} 0 & X_2 \\ 0 & X_4 \end{bmatrix}, \begin{bmatrix} 0 & Y_2 \\ 0 & Y_4 \end{bmatrix} \right) \text{ satisfying (2a), (4a), (6a), and (8a)} \right\}. \end{aligned} \tag{5.3}$$

Moreover, we have $\operatorname{Im} \varphi_1 = \mathscr{S}$ because, obviously, $\operatorname{Im} \varphi_1 \subset \mathscr{S}$ and if

$$\left(\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix} \right) \in \mathscr{S},$$

then

$$\left(\begin{bmatrix} X_1 & 0 \\ X_3 & 0 \end{bmatrix}, \begin{bmatrix} Y_1 & 0 \\ Y_3 & 0 \end{bmatrix} \right) \in R_1$$

and

$$\varphi_i \left(\begin{bmatrix} X_1 & 0 \\ X_3 & 0 \end{bmatrix}, \begin{bmatrix} Y_1 & 0 \\ Y_3 & 0 \end{bmatrix} \right) = \left(\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix} \right),$$

and therefore we have $\mathscr{S} \subset \operatorname{Im} \varphi_1$.

We also have that Im $\varphi_2 \subset \mathscr{S} =$ Im φ_1. On the other hand, it is known that

$$\dim \operatorname{Ker} \varphi_i + \dim \operatorname{Im} \varphi_i = \dim R_i, \qquad I = 1, 2,$$

and thus, from (5.2) and (5.3) we deduce

$$\dim \operatorname{Im} \varphi_1 = \dim \operatorname{Im} \varphi_2,$$

and, since Im $\varphi_2 \subset$ Im φ_1, we have

$$\operatorname{Im} \varphi_2 = \operatorname{Im} \varphi_1. \tag{5.4}$$

Let us observe finally that

$$\left(\begin{bmatrix} I_{q_1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \right) \in R_1$$

and

$$\varphi_1 \left(\begin{bmatrix} I_{q_1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \right) = \left(\begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix}, \begin{bmatrix} I_{p_1} \\ 0 \end{bmatrix} \right).$$

Then, by (5.4), there must exist a pair in R_2 such that its image by φ_2 is

$$\left(\begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix}, \begin{bmatrix} I_{p_1} \\ 0 \end{bmatrix} \right).$$

Thus, this pair of R_2 will have the form

$$\left(\begin{bmatrix} I_{q_1} & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} I_{p_1} & * \\ 0 & * \end{bmatrix} \right).$$

Therefore, we have found a pair in R_2 such that $Y_1 = I_{p_1}$. ∎

Note added in proof. The authors observed that Theorem 2.3 of V. L. Syrmos and F. L. Lewis [17] is equivalent to Theorem 5.1 (Extension of Roth's criterion). On September 27, 1994 Professor Harald K. Wimmer kindly sent the authors a copy of his paper [18] where Theorem 1.1 coincides with Theorem 5.1 mentioned above.

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REFERENCES

1 M. A. Beitia, J. M. Gracia, and I. de Hoyos, A linear matrix equation: A criterion for block similarity, *Linear and Multilinear Algebra* 31:93–118 (1992).

2 W. Brandenbusch, Die Anzahl linear unabhängiger Matrizen X, die mit einer bestimmten Matrix A kommutieren, ausgedrückt in den Weyrschen Charakteristiken, *Z. Angew. Math. Mech.* 60:205 (1980).

3 H. Flnaders and H. K. Wimmer, On the matrix equations $AX - XB = C$ and $AX - YB = C$, *SIAM J. Appl. Math.* 32(4):707–710 (1977).

4 S. Friedland, Analytic similarity of matrices, In *Lectures in Appl. Math.* 18, Amer. Math. Soc., 1980, pp. 43–85.

5 G. Frobenius, Über die mit einer Matrix vertanschbaren Matrize, *Berlin Sitzb.*, 1910, pp. 3–15.

6 F. R. Gantmacher, *Théorie des Matrices*, Vol. 1, Dunod, Paris, 1966.

7 F. R. Gantmacher, *Théorie des Matrices*, Vol. 2, Dunod, Paris, 1966.

8 I. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley, New York, 1986.

9 I. de Hoyos, Points of continuity of the Kronecker canonical form, *SIAM J. Matrix Anal. Appl.* 11(2):278–300 (1990).

10 I. de Hoyos, Perturbación de Matrices Rectangulares y Haces de Matrices, Doctoral Thesis, Universidad del País Vasco, Bilbao, 1990.

11 N. Karcanias and G. Kalogeropoulos, Right, left characteristic sequences and column, row minimal indices of a singular pencil, *Internat. J. Control* 47(4):937–946 (1988).

12 P. Lancaster and M. Tismenetsky, *The Theory of Matrices with Applications*, Academic, London, 1985.

13 B. P. Molinari, Structural invariants of linear multivariable systems, *Internat. J. Control* 28(4):493–510 (1978).

14 A. Pokrzywa, On perturbations and the equivalence orbit of a matrix pencil, *Linear Algebra Appl.* 82:99–121 (1986).

15 G. W. Stewart and J. G. Sun, *Matrix Perturbation Theory*, Academic, New York, 1990.

16 J. S. Thorp, The singular pencil of a linear dynamical system, *Internat. J. Control* 18(3):577–596 (1973).

17 V. L. Syrmos, F. L. Lewis, Decomposability and Quotient Subspaces for the Pencil sL-M, *SIAM J: Matrix Anal. Appl.* 15(3):865–880 (1994).

18 H. K. Wimmer, Consistency of a Pair of Generalized Sylvester Equations, IEEE Trans. Automatic Control 39(5):1014–1016 (1994).

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